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Spectra of Weighted Composition Operators with Quadratic Symbols

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Abstract

Previously, spectra of certain weighted composition operators $W_{\psi,\varphi}$ on H^2 were determined under one of two hypotheses: either φ converges under iteration to the Denjoy-Wolff point uniformly on all of $\mathbb{D}$ rather than simply on compact subsets, or φ is "essentially linear fractional." We show that if φ is a quadratic self-map of $\mathbb{D}$ of parabolic type, then the spectrum of $W_{\psi,\varphi}$ can be found when these maps exhibit both of the aforementioned properties, and we determine which symbols do so.

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1. Introduction

Let H^2 denote the classical *Hardy space*, the Hilbert space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the open unit disk $\mathbb{D}$ such that

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

A *composition operator* C_φ on H^2 is given by $C_\varphi f = f \circ \varphi$. We call φ the *symbol* of the associated composition operator. When φ is an analytic self-map of $\mathbb{D}$, the operator C_φ is bounded. Composition operators on H^2 have been extensively studied for several decades; [1] and [2] are seminal books on the subject. One reason for their study is the deep connection

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to multiplication operators. H^∞, the space of bounded on analytic functions on $\mathbb{D}$, is the multiplier algebra of H^2: if $\psi \in H^\infty$ and $f \in H^2$, then $\psi f \in H^2$. Thus, for any $\psi \in H^\infty$, the *multiplication operator* T_ψ on H^2 is given by $T_\psi f = \psi f$, and all such operators are bounded. Throughout this paper, we will always assume $\psi \in H^\infty$. We write $W_{\psi,\varphi} := T_\psi C_\varphi$ and call such an operator a *weighted composition operator*.

The *spectrum* of an operator T on a Hilbert space H, denoted $\sigma(T)$, is given by $\{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$. A thorough treatment of the spectrum of composition operators on H^2 is given in [1, Chapter 7]. Determining the spectra of *weighted* composition operators on H^2 is still largely an open question, and some results are given in [3] and [4]; in both papers, the results depend on the behavior of φ on $\mathbb{D}$.

Building on the work of [3] and [4], our goal is to answer the following:

If φ is a quadratic self-map of $\mathbb{D}$ and $\psi \in H^\infty$, can we determine $\sigma(W_{\psi,\varphi})$?

While we are interested in quadratic maps, this is not the only way we will classify our choice of symbols. The Denjoy-Wolff theorem guarantees that any analytic self-map of $\mathbb{D}$ (apart from elliptic automorphisms) will have a unique attracting fixed point w in $\overline{\mathbb{D}}$, and that φ converges, uniformly on compact subsets of $\mathbb{D}$, to that point under iteration. This behavior generally splits into three categories:

- w is properly in $\mathbb{D}$ with $|\varphi'(w)| < 1$ (elliptic),
- w is on the boundary and $\varphi'(w) < 1$ (hyperbolic), or
- w is on the boundary and $\varphi'(w) = 1$ (parabolic).

If φ is not analytic on $\partial\mathbb{D}$, the boundary fixed point cases need to be stated more carefully (see [1]), but that will not be an issue for our quadratic symbols.

While our primary goal is to determine $\sigma(W_{\psi,\varphi})$, a secondary goal of ours is to show the interplay between the conditions imposed on φ in [3] and [4]. In the rest of the introduction, we will explain the concept of *uniformly convergent iteration* found in [3], use it to narrow down which quadratic maps we consider, and also explain the concept of *essentially linear fractional* maps found in [4].

1.1. Uniformly convergent iteration.

Let $\varphi_n = \varphi \circ \varphi \cdots \circ \varphi$ (n times). An analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ with Denjoy-Wolff point w exhibits *uniformly convergent iteration* (UCI) if $\varphi_n \to w$ uniformly on *all* of $\mathbb{D}$, rather than simply on compact subsets of $\mathbb{D}$. The main result we need is the following:

Theorem 1.1. *[3, Corollary 10] Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic with Denjoy-Wolff point w, $\varphi_n \to w$ uniformly in $\mathbb{D}$, and $\psi \in H^\infty$ is continuous at $z = w$ with $\psi(w) \neq 0$. Then we have*

$$\overline{\sigma_p(\psi(w)C_\varphi)} \subseteq \overline{\sigma_{ap}(W_{\psi,\varphi})} \subseteq \sigma(W_{\psi,\varphi}) \subseteq \sigma(\psi(w)C_\varphi)$$

In particular, if $\overline{\sigma_p(C_\varphi)} = \sigma(C_\varphi)$, then $\sigma(T_\psi C_\varphi) = \sigma(\psi(w)C_\varphi)$.

If we can show that our quadratic maps exhibit UCI, then Theorem 1.1 gives us $\sigma(W_{\psi,\varphi}) \subseteq \sigma(\psi(w)C_\varphi)$. However, we get more than that. If the Denjoy-Wolff point w is on $\partial\mathbb{D}$ and $\varphi'(w) < 1$, we know that $\overline{\sigma_p(C_\varphi)} = \sigma(C_\varphi)$ [1, Theorem 7.26], and in conjunction with Theorem 1.1, we have

$$\sigma(W_{\psi,\varphi}) = \sigma\left(\psi(w)C_\varphi\right).$$

Furthermore, [3, Theorem 4] gives a clear sufficient condition to check for when φ is UCI if w is on $\partial\mathbb{D}$ and $\varphi'(w) < 1$, so there is no real work to do with quadratic maps. Likewise, if w is properly in $\mathbb{D}$ and φ is UCI, then $W_{\psi,\varphi}$ is power-compact [3, Corollary 2], and again we have $\sigma(W_{\psi,\varphi}) = \sigma(\psi(w)C_\varphi)$ by Theorem 1.1. Therefore, we are really only interested in parabolic maps (w is on the boundary with $\varphi'(w) = 1$).

1.2. φ is of parabolic type

For the rest of the paper, we assume that φ is analytic in $\overline{\mathbb{D}}$ with a single fixed point w on the boundary, given by a quadratic polynomial, and $\varphi'(w) = 1$. Since $C_{e^{i\theta}z}$ is a unitary operator, we can conjugate C_φ by $C_{e^{i\theta}z}$ to rotate the fixed point without affecting the spectrum; therefore we may assume that $w = 1$. We can then simplify the generic quadratic $\varphi(z) = a_2z^2 + a_1z + a$ by noting

$$\begin{aligned} a_2 + a_1 + a &= 1 \\ 2a_2 + a_1 &= 1 \end{aligned}$$

which gives $a_1 = 1 - 2a_2$, and $a_2 = a$, so we have

$$\varphi(z) = az^2 + (1-2a)z + a.$$

We now need to confirm exactly when φ maps $\mathbb{D}$ into $\mathbb{D}$.

Proposition 1.2. Suppose $\varphi(z) = az^2 + (1-2a)z + a$ so that $\varphi(1) = \varphi'(1) = 1$. Then φ is a self-map of $\mathbb{D}$ if and only if $|a - \frac{1}{4}| \le \frac{1}{4}$. Furthermore, the only elements of $\partial\mathbb{D}$ that are not mapped by φ into $\mathbb{D}$ are the point $z = 1$ and, in the case where $|a - \frac{1}{4}| = \frac{1}{4}$, the point $z = -1$.

Proof. Note that $\varphi(-1) = 4a - 1$, so we require that $|4a - 1| \le 1$ for φ to be a self-map of $\mathbb{D}$. Equivalently, we have $|a - \frac{1}{4}| \le \frac{1}{4}$, so that a is contained in the disk of radius $\frac{1}{4}$ centered at $\frac{1}{4}$.

Now suppose instead that $|a - \frac{1}{4}| \le \frac{1}{4}$ is given. Then, we will find the image of the unit circle under φ. Thinking of φ as a function of a, we know by the Maximum Modulus Principle that $|f(z)|$ will be maximized when $|a - \frac{1}{4}| = \frac{1}{4}$. Therefore, writing $a = A + Bi$, we have $A^2 + B^2 = \frac{1}{2}A$, and if $z = X + Yi$, we have $X^2 + Y^2 = 1$.

If $\varphi(z) = az^2 + (1-2a)z + a = (A+Bi)(X+Yi)^2 + (1-2A-2Bi)(X+Yi) + (A+Bi)$, then $|\varphi(z)|^2$ can be directly computed, resulting in a 28-term algebraic expression we will not force the reader to endure. However, using the aforementioned equations, we are able to simplify the expression to $2AX^2 - 2A + 1$. This is a real-valued quadratic in X defined on

$[-1,1]$, with vertex $(0,-2A+1)$ (which has a nonnegative y-value for $0 \leq A \leq \frac{1}{2}$, as needed). Since the graph is a parabola whose defining equation has a positive leading coefficient, the maximum values in the domain occur at the endpoints $x=1$ and $x=-1$, which both result in $|\varphi(z)|^2 = 1$. Note, however, that equality is always attained at $\varphi(1)=1$, but equality at $x=-1$ only happens for $\varphi(-1) = 4a-1$ when $|a-\frac{1}{4}| = \frac{1}{4}$; otherwise $\varphi(-1)$ is properly in $\mathbb{D}$. In any case, for any other value of X, we have $|\varphi(z)|^2 < 1$, so $\mathbb{D}$ is mapped by φ into $\mathbb{D}$ as desired. □

While the primary purpose of Proposition 1.2 is to discover exactly when φ is a self-map of $\mathbb{D}$, we should also note a few key facts from this result that we will use later. In particular, we see that $\text{Re } a > 0$ and $|a| \leq \frac{1}{2}$. Furthermore, we will take advantage of knowing exactly which points on the unit circle are mapped by φ back onto the unit circle. As we have seen in this proof, $|a-\frac{1}{4}| < \frac{1}{4}$ is a separate case from $|a-\frac{1}{4}| = \frac{1}{4}$. This will remain true throughout the paper.

1.3. Essentially linear fractional.

First we give the definition of *essenitally linear fractional* found in [4]:

Definition 1.3 ([4])**.** An analytic self-map φ of $\mathbb{D}$ is *essentially linear fractional* if all of the following hold:

1. $\varphi(\mathbb{D})$ is contained in a proper subsdisk of $\mathbb{D}$ internally tangent to the unit circle at some point $\eta \in \partial\mathbb{D}$;
2. $\varphi^{-1}(\{\eta\}) := \{\gamma \in \partial\mathbb{D} : \eta$ belongs to the cluster set of η of φ at $\gamma\}$ consists of one element, say $\zeta \in \partial\mathbb{D}$; and
3. φ''' extends continuously to $\mathbb{D} \cup \{\zeta\}$.

We are interested in these maps because of certain spectral results in [4]. The essential spectrum $\sigma_e(T)$ is the spectrum of T in the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$, the bounded operators on a Hilbert space H modulo the compact operators. The essential spectrum is always a subset of the spectrum. In particular, we will find symbols where $\sigma_e(W_{\psi,\varphi}) = \sigma_e(\psi(w)C_\varphi)$, and combined with Theorem 1.1, we will arrive at the main theorem of the paper:

Main Theorem. *Suppose $\varphi(z) = az^2 + (1-2a)z + a$ maps $\mathbb{D}$ into $\mathbb{D}$ and $|a-\frac{1}{4}| < \frac{1}{4}$. Then for any $\psi \in H^\infty$ continuous at 1, we have*

$$\sigma(W_{\psi,\varphi}) = \sigma(\psi(1)C_\varphi) = \{\psi(1)e^{-2at} : t \geq 0\} \cup \{0\}.$$

The rest of the paper proceeds as follows. In Section 2, we show that if $\varphi(z) = az^2 + (1-2a)z + a$ with $|a-\frac{1}{4}| \leq \frac{1}{4}$, then φ exhibits UCI. In Section 3, we determine exactly which of these symbols are essentially linear fractional. In Section 4, we combine these results to arrive at our main theorem, as well as a few other results. In Section 5, we discuss the connection between uniformly convergent iteration and essentially linear fractional maps, and propose questions for further research.

2. φ converges uniformly under iteration on all of $\mathbb{D}$ (UCI)

Rather than iterating our quadratic maps directly, we will circumvent this issue by taking alternate approaches to showing that $\varphi_n \to 1$ uniformly on all of $\mathbb{D}$. For $|a - \frac{1}{4}| < \frac{1}{4}$, we rely on arguments from complex dynamics. When $|a - \frac{1}{4}| = \frac{1}{4}$, we work directly with the following alternate characterization of uniform convergence.

Proposition 2.1. Let φ be an analytic self-map of $\mathbb{D}$. Then φ_n converges uniformly on $\mathbb{D}$ to its Denjoy-Wolff point w if and only if

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |\varphi_n(z) - w| \to 0.$$

Theorem 2.2. *Suppose $\varphi(z) = az^2 + (1-2a)z + a$ and $|a - \frac{1}{4}| = \frac{1}{4}$, $a \neq 0$. Then the iterates of φ converge uniformly to 1 on the entire open disk $\mathbb{D}$.*

Proof. We will find a recursive pattern for $\sup_{z\in\mathbb{D}} |\varphi_n(z) - 1|$ that approaches 0.

Suppose $\sup_{z\in\mathbb{D}} |\varphi_n(z) - 1| = r_n$ (the value depends on n) and consider $|\varphi_{n+1}(z) - 1|$. This factors as

$$|a|\,|\varphi_n(z) - 1| \left|\varphi_n(z) - \left(1 - \frac{1}{a}\right)\right|.$$

Since φ_n is an analytic map of $\overline{\mathbb{D}}$, we know that $\sup_{z\in\mathbb{D}} |\varphi_n(z) - 1| = r_n$ is attained at some point z_1 in $\overline{\mathbb{D}}$, which also lies on the circle with center 1 and radius r_n (call this circle C). We wish to find an upper bound on the distance from z_1 to $z_0 = 1 - \frac{1}{a}$. To do this, we also need to find the image of the circle $|a - \frac{1}{4}| = \frac{1}{4}$ (ignoring $a = 0$) under this transformation. Again, if $a = A + Bi$, then $A^2 + B^2 = \frac{1}{2}A$. Then we have

$$\frac{1}{a} = \frac{1}{A+Bi} = \frac{A - Bi}{A^2 + B^2} = \frac{2(A - Bi)}{A} = 2 - \frac{B}{A}i,$$

so we have $z_0 = 1 - \frac{1}{a} = -1 + \frac{B}{A}i$, and we know $0 < A \leq \frac{1}{2}$. Without loss of generality, assume B is positive. Then the furthest point from C, still within $\overline{\mathbb{D}}$, from z_0 is the point where C intersects the unit circle in the fourth quadrant (see Figure 1). Writing $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = r_n^2$, we can write z_1 in terms of r_n: $(1 - r_n^2/2) - \left(r_n\sqrt{1 - \frac{1}{4}r_n^2}\right) i$. We now wish to find $|z_1 - z_0|$:

$$\left|\varphi_n(z) - \left(1 - \frac{1}{a}\right)\right| \leq |z_1 - z_0| = \sqrt{1 + 2r_n^2 + \frac{B^2}{A^2} - \frac{2B}{A} r_n \sqrt{1 - \frac{1}{4}r_n^2}}.$$

To incorporate the rest of our original expression for $|\varphi_{n+1}(z) - 1|$, we recall that $|a| = \sqrt{A^2 + B^2} = \sqrt{\frac{1}{2}A}$ and $|\varphi_n(z) - 1| = r_n$. We can also substitute $B = \sqrt{\frac{1}{2}A - A^2}$ to get

$$\sup_{z\in\mathbb{D}} |\varphi_{n+1}(z) - 1| \leq r_n \sqrt{r_n^2 A + \frac{1}{4} - r_n \sqrt{\left(\frac{1}{2}A - A^2\right)\left(1 - \frac{1}{4}r_n^2\right)}}.$$

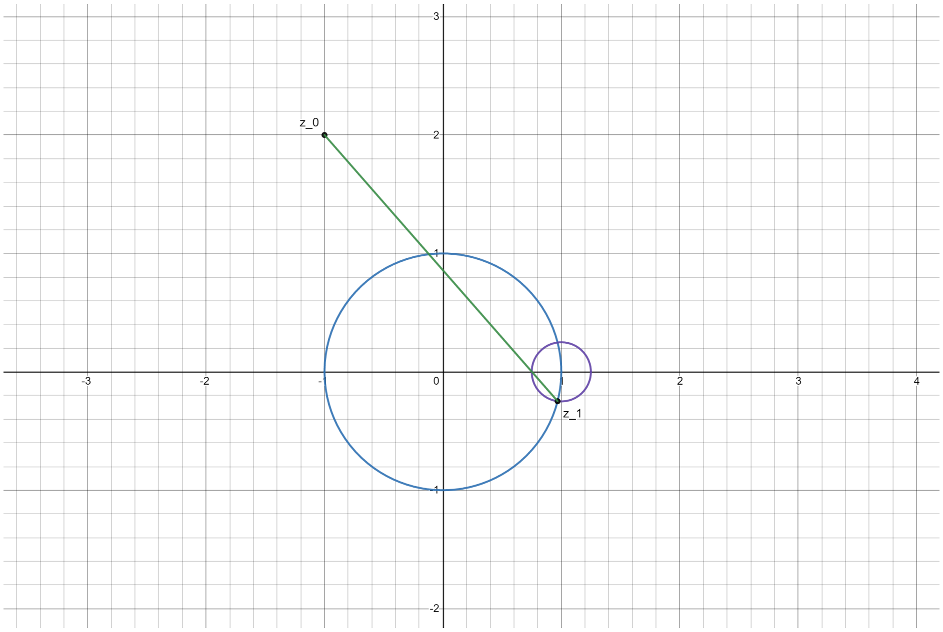


Figure 1: An example scenario for z_1 and z_0.

For our base case, consider $r_1 = |\varphi(z) - 1|$. Then for $z = X + Yi \in \partial\mathbb{D}$, if $a = A + Bi$, by direct computation we have

$$|\varphi(z) - 1|^2 = 2 - 2AX^2 + 4AX - 2A + 4(X-1)\sqrt{A/2 - A^2}\sqrt{1 - X^2} - 2X.$$

For smaller values of A, this value is maximized at $X = -1$. However, when $|(4a-1)-1|^2 = |4A - 2 + 4Bi|^2 = 1$, we have $A = \frac{3}{8}$, and the situation changes. For $\frac{3}{8} \leq A \leq \frac{1}{2}$, we have $|\varphi(z) - 1| \leq 1$ for all $z \in \overline{\mathbb{D}}$. For $0 < A < \frac{3}{8}$, the maximum value of $|\varphi(z) - 1|$ is $2\sqrt{1-2A}$, so $r_1 = \max\{2\sqrt{1-2A}, 1\}$.

For $0 < A \leq \frac{1}{2}$, define

$$f(x) = \sqrt{x^2 A + \frac{1}{4} - x\sqrt{\left(\frac{1}{2}A - A^2\right)\left(1 - \frac{1}{4}x^2\right)}},$$

so f satisfies $0 < f(x) < 1$ for $0 \leq x \leq \max\{2\sqrt{1-2A}, 1\}$. Then

$$r_{n+1} = r_n\sqrt{r_n^2 A + \frac{1}{4} - r_n\sqrt{\left(\frac{1}{2}A - A^2\right)\left(1 - \frac{1}{4}r_n^2\right)}} = r_n f(r_n) < r_n,$$

and thus, our sequence is decreasing. Since r_1 is positive and $0 < f(r_n) < 1$ for all $n \geq 1$, we know that the sequence r_n is bounded below by 0. Since our sequence is monotonically decreasing and bounded below, it converges. Then $c = \lim r_{n+1} = \lim r_n$, which means that $c = cf(c)$ where f is defined as above. We have already established that f is always positive, so the only solution is $c = 0$. Therefore, $\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |\varphi_n(z) - 1| = 0$, so φ_n converges under iteration to 1 uniformly on *all* of $\mathbb{D}$.

□

For $|a - \frac{1}{4}| < \frac{1}{4}$, we now turn to traditional results in complex dynamics, via Beardon [5].

Theorem 2.3. *Suppose* $\varphi(z) = az^2 + (1-2a)z + a$ *and* $|a - \frac{1}{4}| < \frac{1}{4}$. *Then the iterates of* φ *converge uniformly to* 1 *on the entire open disk* $\mathbb{D}$.

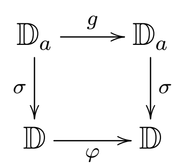
Proof. Consider the family of degree two polynomial maps $\varphi\colon \mathbb{D} \to \mathbb{D}$ given by

$$\varphi(z) = az^2 + (1-2a)z + a,$$

where $|a - 1/4| < 1/4$. These maps all have a parabolic fixed point at $z = 1$ and, as maps of $\mathbb{C}$, are conjugate to the map $z \mapsto z - z^2$. Specifically, for the maps $g, \sigma\colon \mathbb{C} \to \mathbb{C}$ given by

$$\begin{aligned} g(z) &= z - z^2 \quad \text{and} \\ \sigma(z) &= -\frac{1}{a}z + 1, \end{aligned}$$

we have $g = \sigma^{-1} \circ \varphi \circ \sigma$ on $\mathbb{C}$. Since φ is forward invariant on the disk, g is forward invariant on $\mathbb{D}_a := \sigma^{-1}(\mathbb{D})$, and we have the following commutative diagram:



In particular, φ_n converges uniformly to 1 on $\mathbb{D}$ if g_n converges uniformly to 0 on $\mathbb{D}_a$.

For specific details regarding the dynamics of one complex variable, we refer the reader to ([5, 6, 7]). The domain of g is partitioned into two totally invariant sets, the Julia set, denoted $J(g)$, and the Fatou set, denoted $\mathcal{F}(g)$. The Fatou set is the set of points for which the sequence of iterates forms a normal family, and the Julia set is the complement of the Fatou set. In this case, $\mathcal{F}(g)$ is just the disjoint union of the two open sets:

$$\begin{aligned} B(g,0) &= \{z \in \mathbb{C}\colon g_n(z) \to 0\} \quad \text{and} \\ B(g,\infty) &= \{z \in \mathbb{C}\colon g_n(z) \to \infty\}, \end{aligned}$$

called the basin of zero and the basin of infinity, respectively. The set $J(g)$ has no interior, so it must be that $\mathbb{D}_a$ is a subset of $B(g,0)$ or $B(g,\infty)$. Note that $\sigma^{-1}(0) = a$, and we have assumed that $|a - 1/4| < 1/4$. Define

$$A = \{a \in \mathbb{C}\colon |a - 1/4| < 1/4\}.$$

Since

$$\left|g(a) - \frac{1}{4}\right| = \left|a - a^2 - \frac{1}{4}\right| = \left|a - \frac{1}{2}\right|^2 \le \left(\left|a - \frac{1}{4}\right| + \frac{1}{4}\right)^2 < \frac{1}{4},$$

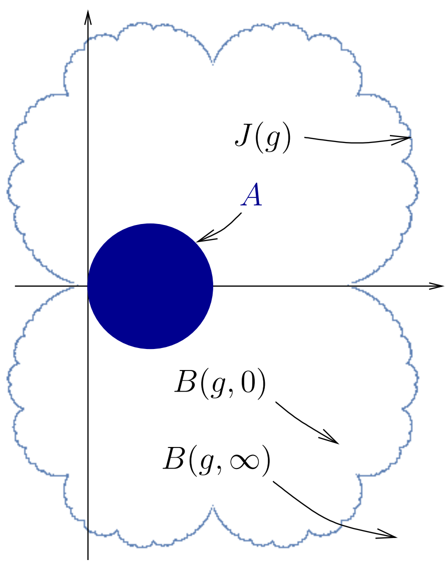


Figure 2: $B(g, 0)$ contains A.

we have that A, an open set with nonempty interior, is forward invariant by g, so A is a subset of either $B(g, 0)$ or $B(g, \infty)$. It is easily verified that for real $a \in A$, $g_n(a) \to 0$. Thus, we have $A \subset B(g, 0)$ (see Figure 2), so $\sigma^{-1}(0) \in B(g, 0)$ as well. It follows that $\mathbb{D}_a \subset B(g, 0)$, and g_n converges uniformly on compact subsets of $B(g, 0)$.

We complete the proof by adapting the proof of the Petal Theorem from [5], for which we need one more conjugacy. Observe that g is conjugate by $\sigma_0(z) = 1/z$ to $h \colon \mathbb{C} \to \mathbb{C}$ given by

$$h(z) = z + 1 + \frac{1}{z - 1}.$$

The fixed point $z = 0$ for g corresponds to the fixed point at ∞ for h. It is easy to show that the half plane $\{z \in \mathbb{C} \colon \operatorname{Re} z > 3\}$ is forward invariant by h, so it is contained in $B(h, \infty)$. The image of this set by σ_0 is a disk of radius $1/6$ centered at $1/6$, contained in $B(g, 0)$ (since the original set was contained in $B(h, \infty)$). However, based on the picture of this basin in Figure 2, it appears we can construct a much larger forward invariant set.

Instead of starting with a half plane for h, we will use the following parabolic region. For each $t \geq 0$, let

$$\Pi + t := \{z = x + iy \colon y^2 > 12(3 + t - x)\}.$$

It can be shown that $\Pi := \Pi + 0$ is forward invariant by h. In particular, if $z \in \Pi$, we will show that $h(z) \in \Pi + 1/2$. Let $z = x + iy$, $h(z) = X + iY$, and $1/(z - 1) = u + iv$, so

$X = x + 1 + u$ and $Y = y + v$; then

$$\begin{aligned} Y^2 - 12(3 + 1/2 - X) &= (y + v)^2 - 12(3 + 1/2 - (x + 1 + u)) \\ &= y^2 - 12(3 + 1/2 - x) + v^2 + 2yv + 12(1 + u) \\ &> v^2 + 2yv + 12(1 + u) \\ &\geq 2yv + 12(1 + u) \\ &\geq 12 - 2|yv| - 12|u| \\ &> 0, \end{aligned}$$

where the last inequality can be derived from the fact that $z \in \Pi$ implies $|z| > 3$. Since $Y^2 > 12(3+1/2-X)$, we have $h(z) \in \Pi + 1/2$. We also have inductively that if $z \in \Pi$, then for all positive integers n,

$$h_n(z) \in \Pi + \frac{n}{2}.$$

Thus, for $h_n(x) + X_n + iY_n$,

$$|h_n(z)|^2 = X_n^2 + Y_n^2 > X_n^2 + 12(2 + n/2 - X_n) = (X_n - 6)^2 + 6n > n,$$

so $|h_n(z)| > \sqrt{n}$ and $h_n(z) \to \infty$ uniformly on Π. The image of Π by σ_0 is the cardioid

$$P := \sigma_0(\Pi) = \{z = re^{i\theta} \colon 6r < 1 + \cos\theta\},$$

and $g_n(z) \to 0$ uniformly on P. The set P is the "petal" referred to in the Petal Theorem.

The preimage of $\mathbb{D}_a$ by σ_0 is the half plane

$$H_a := \sigma_0(\mathbb{D}_a) = \{z = x + iy \colon 2\mathrm{Re}\, a + 2\mathrm{Im}\, a > 1\}.$$

Note that ∂H_a, the boundary of H_a, intersects the x-axis at $x = 1/(2\mathrm{Re}\, a)$, and since $\mathrm{Re}\, a > 0$, it is also never a horizontal line. Moreover, $\partial\Pi$ is a horizontally oriented parabola, intersecting the x-axis at $x = 3$. See Figure 3. Thus, if $1/6 < \mathrm{Re}\, a < 1/4$, then ∂H_a intersects the x-axis at $x < 3$, so ∂H_a must intersect $\partial\Pi$ at exactly two finite points. If $0 < \mathrm{Re}\, a \leq 1/6$, then ∂H_a intersects $\partial\Pi$ at exactly two finite points, one point (at which ∂H_a is tangent to $\partial\Pi$), or zero points. In the last two cases, we have $H_a \subset \Pi$.

Returning to the coordinates centered at zero, this implies that either $\mathbb{D}_a \subset P$ or $\partial\mathbb{D}_a$ intersects ∂P at exactly two nonzero points. Recall that the set $\mathbb{D}_a$ is a disk of radius $|a|$ centered at $\bar{a}$, so both ∂P and $\partial\mathbb{D}_a$ always contains the origin. If $\partial\mathbb{D}_a$ intersects ∂P at exactly two nonzero points, $\overline{\mathbb{D}}_a \backslash P$ is nonempty with a boundary consisting of the curve segment of ∂P between the two nonzero intersections and the curve segment of $\partial\overline{\mathbb{D}}_a$ outside P and between the two nonzero intersections. Thus, $\overline{\mathbb{D}}_a \backslash P$ is closed and bounded, so it is compact. It follows that $\overline{\mathbb{D}}_a \backslash P$ is always a strict, compact (sometimes trivially) subset of $B(g, 0)$.

Since g_n converges uniformly on P and $\overline{\mathbb{D}}_a \backslash P \subset B(g, 0)$, we have that g_n converges uniformly on $\mathbb{D}_a$. Therefore, φ_n converges uniformly on $\mathbb{D}$.

□

Now we have the full picture:

Corollary 2.4. *Suppose $\varphi(z) = az^2 + (1 - 2a)z + a$ and $|a - \frac{1}{4}| \leq \frac{1}{4}$. Then the iterates of φ converge uniformly to 1 on the entire open disk $\mathbb{D}$.*

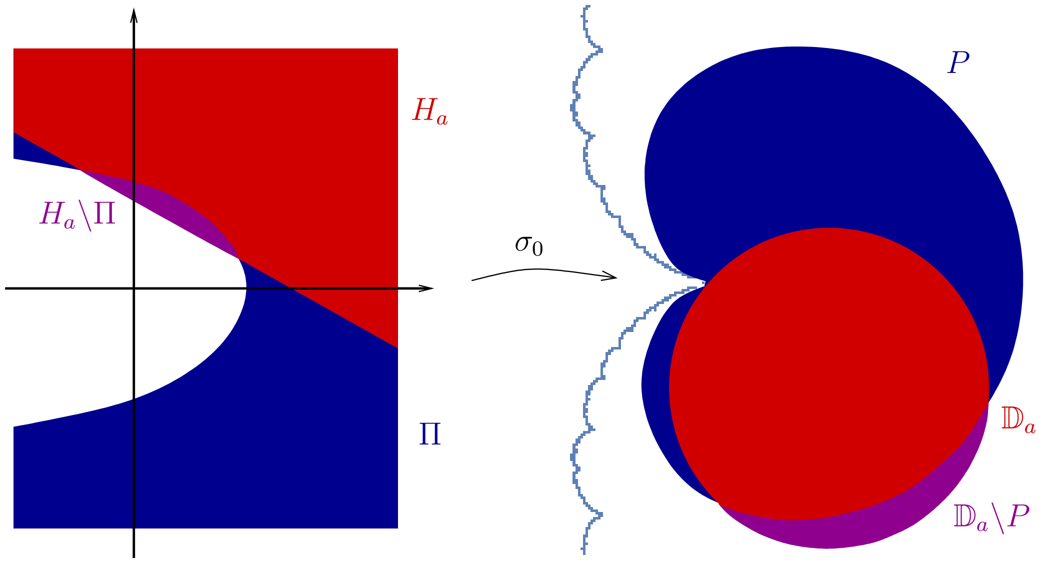


Figure 3: On the left are Π, H, and $H_a \backslash \Pi$. On the right are the images of these sets under σ_0: the cardioid, P, and $\mathbb{D}_a$ with boundaries intersecting at two nonzero points.

3. φ is essentially linear fractional

We have shown that when $\varphi(z) = az^2 + (1-2a)z + a$ with $|a - \frac{1}{4}| \leq \frac{1}{4}$, φ converges uniformly on all of $\mathbb{D}$. Thanks to the work of [3], we know that for any $\psi \in H^\infty$, we have $\sigma(W_{\psi,\varphi}) \subseteq \sigma(\psi(a)C_\varphi)$. We would like for this containment to become equality, and in order to do so, we introduce another property that φ exhibits.

Here we remind the reader of Definition 1.3. An analytic self-map φ of $\mathbb{D}$ is *essentially linear fractional* [4] if all the following hold:

1. $\varphi(\mathbb{D})$ is contained in a proper subsdisk of $\mathbb{D}$ internally tangent to the unit circle at some point $\eta \in \partial\mathbb{D}$;
2. $\varphi^{-1}(\{\eta\}) := \{\gamma \in \partial\mathbb{D} : \eta$ belongs to the cluster set of η of φ at $\gamma\}$ consists of one element, say $\zeta \in \partial\mathbb{D}$; and
3. φ''' extends continuously to $\mathbb{D} \cup \{\zeta\}$.

We will quickly verify that *most* of the the maps $\varphi(z) = az^2 + (1-2a)z + a$ satisfy these conditions. (Again, we assume $a \neq 0$.) To do so, we need another result from [4].

Proposition 3.1 ([4], Proposition 1.3)**.** Let φ be an analytic self-map of $\mathbb{D}$ that extends to be continuous on $\partial\mathbb{D}$. Suppose that $\varphi \in C^2(1)$, that $\varphi(1) = 1$, and that $|\varphi(\zeta)| < 1$ for $\zeta \in \partial\mathbb{D}\backslash\{1\}$. If

$$\operatorname{Re}\left(\frac{1}{\varphi'(1)} - 1 + \frac{\varphi''(1)}{\varphi'(1)^2}\right) > 0 \tag{3.1}$$

then $\varphi(\mathbb{D})$ is contained in a proper subdisk of $\mathbb{D}$ internally tangent to $\partial\mathbb{D}$ at 1.

Now to verify which our family of symbols φ satisfy Definition 1.3:

1. For our symbols, the left side of expression 3.1 simplifies to $2a$, and we know $\operatorname{Re} a > 0$. However, Proposition 3.1 also requires that the rest of the unit circle be mapped into $\mathbb{D}$. If $|a - \frac{1}{4}| < \frac{1}{4}$, Proposition 3.1 applies and $\varphi(\mathbb{D})$ is contained in a subdisk of $\mathbb{D}$ internally tangent at 1. However, we cannot use this proposition if $|a - \frac{1}{4}| = \frac{1}{4}$, since -1 maps to $4a - 1$. If $a \neq \frac{1}{2}$, however, it is worth noting that $\varphi \circ \varphi$ satisfies the definition.
2. Since our function is analytic on the boundary, we are asking that $\varphi^{-1}(\{1\})$ contain only a single point from $\partial\mathbb{D}$. The points that map to 1 are 1 and $1 - \frac{1}{a}$, and the latter is outside of $\overline{\mathbb{D}}$ if $a \neq \frac{1}{2}$. If $a = \frac{1}{2}$, then φ is *not* essentially linear fractional since -1 also maps to 1, and neither is any iterate of φ, so we will handle that case separately.
3. Since $\varphi'''(z) \equiv 0$, this is trivial.

Unsurprisingly, just as with our work on UCI, we see that the definition of essentially linear fractional splits our work into the cases when $|a - \frac{1}{4}| < \frac{1}{4}$ and $|a - \frac{1}{4}| = \frac{1}{4}$. This continues in the following spectral results.

4. Spectrum of $W_{\psi,\varphi}$

We now have all the pieces we need; first we remind the reader of Theorem 1.1 from the introduction:

Theorem 1.1. *[3, Corollary 10]. Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic with Denjoy-Wolff point a, $\varphi_n \to a$ uniformly in $\mathbb{D}$, and $\psi \in H^\infty$ is continuous at $z = a$ with $\psi(a) \neq 0$. Then we have*

$$\overline{\sigma_p(\psi(a)C_\varphi)} \subseteq \overline{\sigma_{ap}(T_\psi C_\varphi)} \subseteq \sigma(T_\psi C_\varphi) \subseteq \sigma(\psi(a)C_\varphi)$$

In particular, if $\overline{\sigma_p(C_\varphi)} = \sigma(C_\varphi)$, then $\sigma(T_\psi C_\varphi) = \sigma(\psi(a)C_\varphi)$.

Composition operators with parabolic symbols are notoriously difficult when it comes to spectral problems, and this situation is no different. We have little information about $\sigma_p(C_\varphi)$; instead we will only use the fact that Theorem 4 gives us $\sigma(W_{\psi,\varphi}) \subseteq \sigma(\psi(a)C_\varphi)$. We now turn to two results from [4] regarding essentially linear fractional maps.

Theorem 4.1. *[4, Theorem 3.3] Suppose that φ is an essentially linear fractional self-map of $\mathbb{D}$ fixing 1. Suppose also that for $s = \varphi''(1)$, Re $s > 0$. Then*

$$\sigma(C_\varphi) = \sigma_e(C_\varphi) = \{e^{-st} : t \geq 0\} \cup \{0\}.$$

Theorem 4.2. *[4, Theorem 4.3] Suppose φ is essentially linear fractional with $\varphi(1) = 1$, and $\psi \in H^\infty$ is continuous at 1. Then $W_{\psi,\varphi} \equiv \psi(1)C_\varphi$ modulo the compact operators.*

Putting these facts together, we arrive at our main theorem.

Theorem 4.3 (Main Theorem)**.** *Suppose $\varphi(z) = az^2 + (1-2a)z + a$ maps $\mathbb{D}$ into $\mathbb{D}$ and $|a - \frac{1}{4}| < \frac{1}{4}$. Then for any $\psi \in H^\infty$ continuous at 1, we have*

$$\sigma(W_{\psi,\varphi}) = \sigma(\psi(1)C_\varphi) = \{\psi(1)e^{-2at} : t \geq 0\} \cup \{0\}.$$

Proof. By Theorem 4, we have $\sigma(W_{\psi,\varphi}) \subseteq \sigma(\psi(1)C_\varphi)$. By Theorem 4.1, we have $\sigma(C_\varphi) = \sigma_e(C_\varphi)$. By Theorem 4.2, we have $\sigma_e(W_{\psi,\varphi}) = \sigma_e(\psi(1)C_\varphi)$. Putting these together, we have

$$\sigma(\psi(1)C_\varphi) = \sigma_e(\psi(1)C_\varphi) = \sigma_e(W_{\psi,\varphi}) \subseteq \sigma(W_{\psi,\varphi}) \subseteq \sigma(\psi(1)C_\varphi),$$

and since the first and last sets in the containment are equal, we have $\sigma(\psi(1)C_\varphi) = \sigma(W_{\psi,\varphi})$. By Theorem 4.1, noting $\varphi''(1) = 2a$, we have

$$\sigma(W_{\psi,\varphi}) = \{\psi(1)e^{-2at} : t \geq 0\} \cup \{0\}.$$

□

We now investigate the situation when $|a - \frac{1}{4}| = \frac{1}{4}$. While we have shown that φ still converges under iteration to 1 uniformly on all of $\mathbb{D}$, it is *not* essentially linear fractional since $|\varphi(-1)| = |4a - 1| = 1$. Here, we must actually divide our special case yet again: if $a \neq \frac{1}{2}$, then $\varphi_2 = \varphi \circ \varphi$ is essentially linear fractional (since $4a - 1$ is then mapped into $\mathbb{D}$) and of course φ_2 is uniformly convergent under iteration on all of $\mathbb{D}$. Therefore, we as a corollary to Theorem 4.3, we get the following:

Corollary 4.4. *Suppose $\varphi(z) = az^2 + (1-2a)z + a$ maps $\mathbb{D}$ into $\mathbb{D}$ and $|a - \frac{1}{4}| = \frac{1}{4}$ for $a \notin \mathbb{R}$. Then for any $\psi \in H^\infty$ continuous at 1, we have*

$$\sigma(W_{\psi,\varphi_2}) = \sigma(\psi(1)C_{\varphi_2}) = \{\psi(1)e^{-4at} : t \geq 0\} \cup \{0\}$$

where $\varphi_2 = \varphi \circ \varphi$.

Proof. The proof follows exactly as before, except that $\varphi_2''(1) = 4a$. □

The result of Corollary 4.4 suggests that it is most likely true that $\sigma(W_{\psi,\varphi})$ is the same as shown in Theorem 4.3 when $|a - \frac{1}{4}| = \frac{1}{4}$, but we do not have a proof.

However, even then, $a = \frac{1}{2}$ proves itself to be an entirely distinct case. Here, instead of functional behavior, we now rely on a linear algebra trick also used in [3].

Lemma 4.5. *[3, Lemma 3] If A and B are bounded linear operators on a Hilbert space $\mathcal{H}$, then $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$.*

Using this, we can now finish the story with $a = \frac{1}{2}$, which actually varies just slightly from the result in Theorem 4.3.

Theorem 4.6. *Suppose $\varphi(z) = \frac{1}{2}z^2 + \frac{1}{2}$, an analytic self-map of $\mathbb{D}$. Then for any $\psi \in H^\infty$ continuous at 1, we have*

$$\sigma(W_{\psi,\varphi}) = \{\psi(1)e^{-t/2} : t \geq 0\} \cup \{0\}.$$

Proof. If $f(z) = \frac{1}{2}z + \frac{1}{2}$, then we have $C_\varphi = C_{z^2}C_f$. By Lemma 4.5, we have $\sigma(C_\varphi) \cup \{0\} = \sigma(C_{z^2}C_f) \cup \{0\} = \sigma(C_f C_{z^2}) \cup \{0\} = \sigma(C_{f^2}) \cup \{0\}$. Note that $f^2(z) = \frac{1}{4}z^2 + \frac{1}{2}z + \frac{1}{4}$, which falls under Theorem 4.1 with $s = \frac{1}{2}$. Since we also know C_φ is not invertible, we have $\sigma(C_\varphi) = \{\psi(1)e^{-t/2} : t \geq 0\} \cup \{0\}$.

Likewise, let $\psi \in H^\infty$ be continuous at 1 and consider $T_\psi C_\varphi = T_\psi C_{z^2} C_f$. Again by Lemma 4.5, we have $\sigma(T_\psi C_{z^2} C_f) \cup \{0\} = \sigma(C_f T_\psi C_{z^2}) \cup \{0\} = \sigma(T_{\psi \circ f} C_{f^2}) \cup \{0\}$. Since f maps $\mathbb{D}$ into $\mathbb{D}$ analytically and f is continuous at 1, we still have that $\psi \circ f \in H^\infty$ and and $\psi \circ f$ is continuous at 1 (and $\psi \circ f(1) = \psi(1)$). Again, we also know that $T_\psi C_\varphi$ is not invertible. Then, by Theorem 4.3 we have

$$\sigma(W_{\psi,\varphi}^*) = \sigma(\psi(1)C_\varphi) = \{\psi(1)e^{-t/2} : t \geq 0\} \cup \{0\}.$$

□

While our guess is that Theorem 4.3 holds for $|a - \frac{1}{4}| = \frac{1}{4}$ when a is complex, the exponent in our result for $a = \frac{1}{2}$ does not match up with Theorem 4.3, presumably because it is more distinct in its failure to be essentially linear fractional. However, the final result is the same in practicality: for $0 < a \leq \frac{1}{2}$, $\sigma(W_{\psi,\varphi})$ is the closed line segment connecting $\psi(1)$ to the origin.

5. Implications and Further Questions

There are two important concepts in this paper that could be pursued further. The first is extending our methods for showing quadratics exhibit uniformly convergent iteration to higher-degree polynomials. Certainly Corollary 2.4 implies more than it says; e.g. $\varphi \circ \varphi$ is a quartic self-map of $\mathbb{D}$ that also converges uniformly on all of $\mathbb{D}$. For any polynomial map fixing 1, $\varphi_n(z) - 1$ will be a factor of $\varphi_{n+1}(z) - 1$, suggesting that our recursive approach used in Theorem 2.2 could be generalized, but it will require deeper geometric intuition than we use here for quadratic maps.

The second important concept is the intersection of self-maps of $\mathbb{D}$ that exhibit uniformly convergent iteration on all of $\mathbb{D}$, and essentially linear fractional maps. Certainly they do not perfectly align; we have already seen that $\frac{1}{2}z^2 + \frac{1}{2}$ converges under iteration to 1 uniformly on all of $\mathbb{D}$, but is not essentially linear fractional. Likewise, a linear fractional map with both an interior and a boundary fixed point (e.g. $\varphi(z) = \frac{z}{2-z}$) cannot converge uniformly on all of $\mathbb{D}$; it must have only one fixed point in $\overline{\mathbb{D}}$ [3, Theorem 3]. However, the intersection of the two concepts is non-trivial, and leads to the following conjecture.

Conjecture 5.1. *Suppose φ is an essentially linear fractional self-map of $\mathbb{D}$ with exactly one fixed point w in $\overline{\mathbb{D}}$. Then φ converges under iteration to w uniformly on all of $\mathbb{D}$.*

Were this conjecture true, it would immediately provide a full description of $\sigma(W_{\psi,\varphi})$ for a rather broad class of symbols, by the same arguments made in this paper. Thus we end with the following list of questions:

1. What is $\sigma(W_{\psi,\varphi})$ if $\varphi(z) = az^2 + (1-2a)z + a$ and $|a - \frac{1}{4}| = \frac{1}{4}$ for $a \notin \mathbb{R}$?

2. When do essentially linear fractional maps converge uniformly to their Denjoy-Wolff point on all of $\mathbb{D}$?
3. Which polynomial self-maps of $\mathbb{D}$ converge uniformly on all of $\mathbb{D}$ to the Denjoy-Wolff point?
4. If $\psi \in H^\infty$ is continuous at the Denjoy-Wolff point w of φ, and φ is not an automorphism, then when, if ever, is $\sigma(W_{\psi,\varphi}) \neq \sigma(\psi(w)C_\varphi)$?

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