From Perfect Shuffles to Landau's Function

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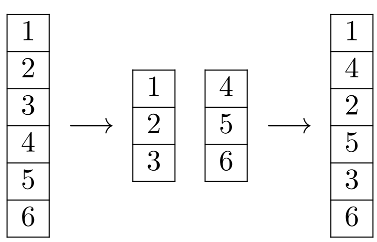
Abstract

If we view a given shuffle of a deck of cards as a permutation, then repeatedly applying this same shuffle will eventually return the deck to its original order. In general, how many steps will that take? What happens in the case of so-called perfect shuffles? What type of shuffle will require the greatest number of applications before restoring the original deck? This paper will address those questions and provide a brief history of the work of Edmund Landau on the maximal order of a permutation in the symmetric group on n objects. It will also note some recent progress in refining his results.

1 Shuffling Cards

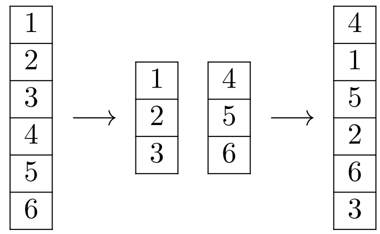
1.1 Perfect Shuffles as Permutations

Given a deck with an even number of cards, we define a *perfect shuffle* as one which splits the deck into two halves and then interlaces them perfectly, as noted in [5] and [18]. Our goal is to view such a shuffle as a permutation, so we start by providing an example with $n = 6$ cards.



Permutation: $p = (1)(2\ 3\ 5\ 4)(6)$

This type of perfect shuffle, known as an *out-shuffle*, leaves the top card on top and the bottom card on bottom. A perfect shuffle which moves the top card to the second position in the deck is an *in-shuffle*. We compare the in-shuffle on $n = 6$ cards with the previous corresponding out-shuffle.



Permutation: $q = (1\ 2\ 4)(3\ 6\ 5)$

We have written these permutations as products of disjoint cycles, in order to view them as elements of the symmetric group $\mathcal{S}_n$ on n objects (where the group operation is composition). In particular, we wish to compute their orders in the group, so we apply the fact that the order is the least common multiple of the cycle lengths. When $n = 6$, the order of the out-shuffle in $\mathcal{S}_6$ is 4, which means that four repeated out-shuffles will return the deck to its original order. Similarly, the order of the in-shuffle is 3, so three repeated in-shuffles will return the deck to its original order. In general, we would like to know the order of an out-shuffle and an in-shuffle in $\mathcal{S}_n$ when n is even.

Even more generally, we also hope to calculate the maximal order of an element in this group and to determine how many elements achieve the maximal order. Equivalently, we seek the type of shuffle of a deck of cards that would take the most repeated applications to restore the original deck order. In the next section, we examine the order structure of $\mathcal{S}_n$ for small values of n and give results for the standard 52-card deck.

1.2 Partitions and Orders for Out-Shuffles and In-Shuffles

In abstract algebra classes, the symmetric group on n objects provides students with a useful early example of a non-abelian group for $n \geq 3$. Since $\mathcal{S}_3$ is the smallest such group, we start by determining its order structure and counting the number of elements which have maximal order. Listing the elements of this group as

$$\mathcal{S}_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\},$$

we may express its order structure in the form 1-1, 3-2, 2-3 (one element of order 1, three elements of order 2, and two elements of order 3). So the maximal order in $\mathcal{S}_3$ is 3, achieved by two elements. Of course, the fact that every element in $\mathcal{S}_3$ is a cycle simplifies these calculations considerably, but $\mathcal{S}_n$ does not have this property for $n \geq 4$.

Accordingly, in moving to an analysis of $\mathcal{S}_4$ and $\mathcal{S}_5$, we note that the orders of elements correspond to partitions into disjoint cycle lengths, so we must apply the least common multiple property. For example, in $\mathcal{S}_5$ the order of any 5-cycle, such as $(2\ 5\ 1\ 4\ 3)$, is 5; similarly, the order of the product of a 3-cycle with a disjoint 2-cycle, such as $(2\ 5\ 1)(4\ 3)$, is 6. In particular, in order to count the number of elements of each possible order in $\mathcal{S}_n$, we must know all the different types of partitions of n. Examining $\mathcal{S}_4$ in detail yields:

Partition of 4	Order in $\mathcal{S}_4$	Number with Order
4	4	$4!/4 = 6$
3+1	3	$4!/3 = 8$
2+2	2	$4!/(2^2 \cdot 2!) = 3$
2+1+1	2	$4!/(2 \cdot 2!) = 6$
1+1+1+1	1	$4!/4! = 1$

Hence the order structure of $\mathcal{S}_4$ is 1-1, 9-2, 8-3, 6-4; in particular, six elements achieve the maximal order of 4.

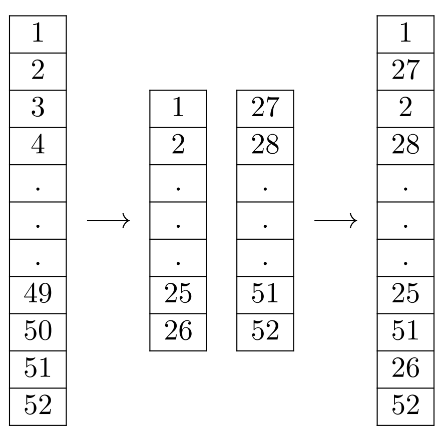
Similarly, analyzing $\mathcal{S}_5$ produces:

Partition of 5	Order in $\mathcal{S}_5$	Number with Order
5	5	$5!/5 = 24$
4+1	4	$5!/4 = 30$
3+2	6	$5!/(3 \cdot 2) = 20$
3+1+1	3	$5!/(3 \cdot 2!) = 20$
2+2+1	2	$5!/(2^2 \cdot 2!) = 15$
2+1+1+1	2	$5!/(2 \cdot 3!) = 10$
1+1+1+1+1	1	$5!/5! = 1$

Hence the order structure of $\mathcal{S}_5$ is 1-1, 25-2, 20-3, 30-4, 24-5, 20-6; in particular, twenty elements achieve the maximal order of 6.

But this approach of listing all possible partitions is certainly not practical for a standard 52-card deck, since 52 may be partitioned in 281,589 ways [20]. Even so, we are able to determine which of these partitions corresponds to an out-shuffle and which corresponds to an in-shuffle, allowing us to calculate the order of both types of perfect shuffles in the corresponding group. Later, we will also find the partitions which correspond to the maximal order as well as calculating the number of elements in $\mathcal{S}_{52}$ with that order.

We start with the out-shuffle on a standard deck of 52 cards:



Permutation: $p = ?$

We represent this out-shuffle as the following permutation on $\{1, 2, \ldots, 52\}$:

$$p(x) = \begin{cases} 2x-1 & \text{if } x \le 26 \\ 2(x-26) & \text{if } x \ge 27 \end{cases}$$

We also note that $p(x) \equiv 2x - 1 \pmod{51}$. Then $p = (1)(52)(18\ 35)uvwxyz$, where

$$u = (2\ 3\ 5\ 9\ 17\ 33\ 14\ 27),$$

$$v = (4\ 7\ 13\ 25\ 49\ 46\ 40\ 28),$$

$$w = (6\ 11\ 21\ 41\ 30\ 8\ 15\ 29),$$

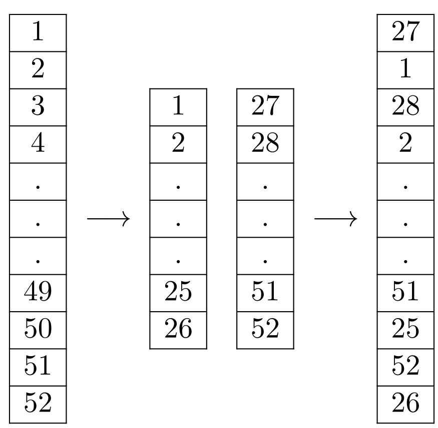
$$x = (10\ 19\ 37\ 22\ 43\ 34\ 16\ 31),$$

$$y = (12\ 23\ 45\ 38\ 24\ 47\ 42\ 32),$$

$$z = (20\ 39\ 26\ 51\ 50\ 48\ 44\ 36).$$

Thus the order of p in $\mathcal{S}_{52}$ is 8. This means that a skilled card shark who is able to perform an out-shuffle each time without any errors will restore the deck to its original order after only eight of these shuffles.

Next, we compare the in-shuffle on 52 cards and compute its order:



Permutation: $q = \ ?$

We may represent this in-shuffle by $q(x) \equiv 2x \pmod{53}$, or by

$$q(x) = \begin{cases} 2x & \text{if } x \le 26 \\ 2(x-26)-1 & \text{if } x \ge 27 \end{cases}.$$

Then $q = (1\ 2\ 4\ 8\ 16\ 32\ 11\ 22\ 44\ 35\ 17\ 34\ 15\ 30\ 7\ 14\ 28\ 3\ 6\ 12\ 24\ 48\ 43\ 33\ 13\ 26\ 52\ 51\ 49\ 45\ 37\ 21\ 42\ 31\ 9\ 18\ 36\ 19\ 38\ 23\ 46\ 39\ 25\ 50\ 47\ 41\ 29\ 5\ 10\ 20\ 40\ 27)$.

Thus the order of q in $\mathcal{S}_{52}$ is 52. Our card shark would find it more challenging to make 52 consecutive in-shuffles to restore the deck.

In general, Diaconis, Graham, and Kantor have noted and proved the following "well known" results [5].

Theorem 1. Given a deck of $n = 2m$ cards:

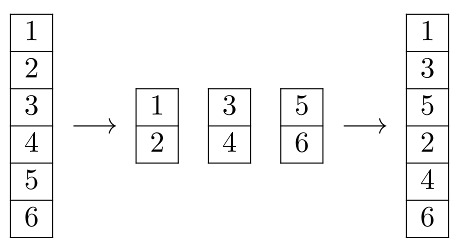
(i) The order of an out-shuffle equals the order of 2 modulo $2m - 1$.

(ii) The order of an in-shuffle equals the order of 2 modulo $2m + 1$.

(iii) The order of an in-shuffle in $\mathcal{S}_n$ is the same as the order of an out-shuffle in $\mathcal{S}_{n+2}$.

We use this theorem to verify the results in our previous examples. For $n = 6$, we note that the order of 2 modulo 5 is 4, since the powers of 2 modulo 5 cycle as 2, 4, 3, 1, ... ; similarly, the order of 2 modulo 7 is 3, as the powers of 2 modulo 7 cycle as 2, 4, 1, ... For $n = 52$, we leave it as an exercise for the reader to confirm that the order of 2 modulo 51 is 8, while the order of 2 modulo 53 is 52.

In an attempt to generalize the notion of a perfect shuffle, we offer the following question: Given a deck with $n = 3m$ cards, how would we define a *perfect 3-shuffle*? The natural approach for such a shuffle would be to divide the deck into three equal piles and then interlace them perfectly. Here is one example with $n = 6$ cards:



Permutation: $r = (1)(2\ 4\ 5\ 3)(6)$

We note that for each deck with $n = 3m$ cards, there are $3! = 6$ perfect 3-shuffles, as opposed to just two perfect 2-shuffles. How do the orders of these six permutations compare? Given an element σ in $\mathcal{S}_3$, we define a σ-shuffle to be the perfect 3-shuffle in which the order of interlacing the three piles of cards follows the order of the elements as listed in σ. We calculate those orders for several small values of n.

n	(1 2 3)	(1 3 2)	(2 1 3)	(2 3 1)	(3 1 2)	(3 2 1)
3	1	2	2	3	3	2
6	4	4	4	6	6	6
9	2	4	4	6	6	4
12	5	21	21	11	11	3
15	6	30	30	15	15	4
18	16	90	90	52	52	18
21	4	24	24	38	38	5
24	11	48	48	51	51	20
27	3	6	6	9	9	6
30	28	28	28	360	360	30

We offer the following conjectures concerning perfect 3-shuffles:

1. The order of a (1 2 3)-shuffle equals the order of 3 modulo $3m - 1$.
2. The order of a (3 2 1)-shuffle equals the order of 3 modulo $3m + 1$.
3. The order of a (1 3 2)-shuffle equals the order of a (2 1 3)-shuffle.
4. The order of a (2 3 1)-shuffle equals the order of a (3 1 2)-shuffle.

1.3 Maximal Order in $\mathcal{S}_{52}$

If our goal instead is to determine the type of shuffle that would take the most repeated applications to restore a standard deck, then we must find the partition of 52 that maximizes the least common multiple of its part lengths. For example, dividing the deck into two piles, one with 25 cards and the other with 27, and simply cycling the cards in each pile separately yields a permutation with order $\text{lcm}(25, 27) = 675$ in $\mathcal{S}_{52}$. Experimenting further leads to additional partitions in which the parts are prime powers that are pairwise relatively prime, such as:

$$\begin{aligned} 52 = 32 + 9 + 11: &\quad \text{order} = \text{lcm}(32,9,11) = 3168 \\ 52 = 3 + 13 + 17 + 19: &\quad \text{order} = \text{lcm}(3,13,17,19) = 12{,}597 \\ 52 = 1 + 3 + 7 + 11 + 13 + 17: &\quad \text{order} = \text{lcm}(1,3,7,11,13,17) = 51{,}051 \end{aligned}$$

As noted in [11], the maximum order of an element in $\mathcal{S}_{52}$ is 180,180. The partition $52 = 1 + 1 + 1 + 4 + 9 + 5 + 7 + 11 + 13$ corresponds to such an element, as

$$\text{lcm}(1, 1, 1, 4, 9, 5, 7, 11, 13) = 180, 180.$$

We observe that the partitions $52 = 1+2+4+9+5+7+11+13$ and $52 = 3+4+9+5+7+11+13$ also correspond to elements in $\mathcal{S}_{52}$ of order 180,180; in fact, this represents the maximal order in $\mathcal{S}_{49}$ as well, since any partition of the three “extra” cards does not contribute to an increase in the least common multiple.

We conclude this section by calculating the number of elements in $\mathcal{S}_{52}$ with maximal order 180,180. The number of permutations in $\mathcal{S}_{52}$ corresponding to the partition $52 = 1+1+1+4+9+5+7+11+13$ is

$$\frac{52!}{13 \cdot 11 \cdot 7 \cdot 5 \cdot 9 \cdot 4 \cdot 3!} \approx 7.460889 \times 10^{61}.$$

Next, the number corresponding to the partition $52 = 1+2+4+9+5+7+11+13$ is

$$\frac{52!}{13 \cdot 11 \cdot 7 \cdot 5 \cdot 9 \cdot 4 \cdot 2} \approx 2.238267 \times 10^{62}.$$

Finally, the number corresponding to the partition $52 = 3+4+9+5+7+11+13$ is

$$\frac{52!}{13 \cdot 11 \cdot 7 \cdot 5 \cdot 9 \cdot 4 \cdot 3} \approx 1.492178 \times 10^{62}.$$

Hence the number of permutations in $\mathcal{S}_{52}$ having maximal order is

$$\frac{6 \cdot 52!}{13 \cdot 11 \cdot 7 \cdot 5 \cdot 9 \cdot 4 \cdot 3!} \approx 4.476533 \times 10^{62}.$$

We also note that these permutations are rare in $\mathcal{S}_{52}$, as the probability of choosing such a permutation at random is

$$\frac{6}{13 \cdot 11 \cdot 7 \cdot 5 \cdot 9 \cdot 4 \cdot 3!} = \frac{1}{180,180}.$$

These results in $\mathcal{S}_{52}$ provide a transition into the next section, in which we address the same questions for $\mathcal{S}_n$ in general. Our focus will be to summarize the contributions of Edmund Landau and others in the study of what is now known as Landau's function.

2 The Life and Legacy of Landau

2.1 Background

Following the summary found in [17], we offer just a brief outline of the biographical highlights of Edmund Landau. Born in Berlin in 1877, Landau remained in the same city to pursue his entire mathematical education, culminating in his doctorate in 1899 under Georg Frobenius. He taught at the University of Berlin for the next ten years, during which he gave a simpler proof of the Prime Number Theorem. He also established an asymptotic result for the maximal order of an element in $\mathcal{S}_n$, which will be the focus of our next section. In 1909, Landau succeeded Hermann Minkowski at the University of Göttingen. Three years later, he challenged the Fifth Congress of Mathematicians to prove or disprove the following four conjectures concerning prime numbers [19]:

1. Goldbach: Every even integer $n \geq 4$ may be written as the sum of two primes.
2. Twin Prime: There are infinitely many primes p such that $p+2$ is also prime.
3. Legendre: For every integer n, there is a prime p with $n^2 < p < (n+1)^2$.
4. Euler: There are infinitely many primes of the form n^2+1.

All four remain unresolved to this day.

After World War I, both Landau and the University of Göttingen flourished, and many promising young mathematicians received inspiration and influence from Landau as well as his colleagues Felix Klein and David Hilbert [8]. Landau's research focused primarily upon analytic number theory, covering a broad range of topics from the distribution of prime ideals to the Riemann zeta function [12]. But in 1933, due to the increasing pressure of anti-Semitism in Germany, Landau was forced to retire from Göttingen; in fact, one of his own students, Oswald Teichmüller, played a major role in Landau's ouster [2]. After this tragic turn of events, Landau was able to lecture only outside Germany. Reflecting upon one of his visits to England, Hardy and Heilbronn noted in [8], "His enforced retirement must have been a terrible blow to him; it was quite pathetic to see his delight when he found himself again in front of a blackboard in Cambridge, and his sorrow when his opportunity came to an end." Ultimately Landau returned to Germany, and he died of a heart attack in Berlin in 1938 [17].

As previously noted, one of Landau's research interests was studying the maximal order of an element in the symmetric group $\mathcal{S}_n$. In his honor, the function $g(n)$ which gives this maximal order for any positive integer n is now known as Landau's function. Equivalently, as we have seen, $g(n)$ is the maximum least common multiple of the parts of a partition of n; the observation that the parts may be taken to be prime powers was first proved by Landau as well.

Before giving Landau's main result in the next section, we examine $g(n)$ for some small values of n and offer several conjectures.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$g(n)$	1	2	3	4	6	6	12	15	20	30	30	60	60	84	105	140	210	210

By its definition, g is a non-decreasing function, but we note that it is not strictly increasing. For instance, from our earlier card examples, we have $g(49) = g(50) = g(51) = g(52) = 180,180$. Also, g never seems to increase by more than a factor of two for consecutive values of n. As we will observe later in the paper, of the following five conjectures about Landau's function, three are true, one is false, and one remains unresolved.

1. For $n > 15$, $g(n)$ is even.
2. The function g is constant on arbitrarily long intervals.
3. For every n, $g(n+1) \leq 2g(n)$.
4. For infinitely many n, $g(n+1) = 2g(n)$.
5. The function g is strictly increasing on arbitrarily long intervals.

2.2 Landau's Theorem

As noted in [4] and [11], in 1903 Landau proved his famous result concerning $g(n)$, publishing his theorem in [9]. He first established the connection between $g(n)$ and the partitions of n, noting that

$$g(n) = \max\{\text{lcm}(n_1, n_2, \ldots, n_r)\},$$

where this maximum is taken over all partitions $n = n_1 + n_2 + \cdots + n_r$. Landau also showed that without loss of generality, the maximum may be taken over all partitions with prime powers as parts; more precisely, he proved that if $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ for distinct primes p_i with $e_i > 0$, and if $\ell(m) = p_1^{e_1} + p_2^{e_2} + \cdots + p_k^{e_k}$, then

$$g(n) = \max\{m : \ell(m) \leq n\}.$$

Using those observations, Landau was able to describe the asymptotic behavior of $g(n)$ in the following result.

Theorem 2. Using log to denote the natural logarithm, we have $\log g(n) \sim \sqrt{n \log n}$ as n goes to infinity; equivalently,

$$\lim_{n\to\infty} \frac{\log g(n)}{\sqrt{n \log n}} = 1.$$

In [11], Miller provided an alternative proof of Landau's result by applying the following technique. Given a positive integer n, find k such that the sum of the first k primes $2, 3, 5, \ldots, p_k$ is less than or equal to n but would exceed n if p_{k+1} were included. We note that this corresponds to seeking a partition of n using the first k primes, with 1's included as needed. Then let $f(n)$ equal the product of these k primes. For example, $f(18) = 2 \cdot 3 \cdot 5 \cdot 7 = 210 = g(18)$. On the other hand, as Miller observed,

$$f(52) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30,030 < 180,180 = g(52),$$

so at first glance, we would not expect $f(n)$ to keep up with $g(n)$ as n increases. Nevertheless, Miller was able to prove that $\log f(n) \sim \log g(n)$. He then established Landau's result by showing that $\log f(n) \sim \sqrt{n \log n}$.

As we noted earlier in the case $n = 52$, it is interesting to compare the size of $g(n)$ to $n!$, the order of $\mathcal{S}_n$. We also examine the question of how many elements in $\mathcal{S}_n$ can be expected to have maximal order $g(n)$. First of all, using Landau's theorem, we observe that $g(n)$ is very small compared to the order of $\mathcal{S}_n$, since Stirling's formula implies

$$\log n! \sim n \log n - n.$$

Next, we let $h(n)$ be the number of elements in $\mathcal{S}_n$ with order $g(n)$. The following table compares $g(n)$ and $h(n)$ for small values of n.

n	1	2	3	4	5	6	7	8	9
$g(n)$	1	2	3	4	6	6	12	15	20
$h(n)$	1	1	2	6	20	240	420	2688	18144
$n!$	1	2	6	24	120	720	5040	40320	362880

We observe that in some, but not all, cases, $g(n)h(n) = n!$. In fact, not many elements in $\mathcal{S}_n$ have order $g(n)$. More precisely, Erdős and Turán showed in [6] that almost all permutations in $\mathcal{S}_n$ have order k with

$$\log k \sim \frac{1}{2} \log^2 n.$$

2.3 Conjectures for Landau's Function

In this section, we return to the five conjectures previously mentioned, and we note several other results and open problems involving Landau's function. The first three of those five conjectures are true, having been proved by Nicolas [15, 16]:

1. For $n > 15$, $g(n)$ is even.
2. The function g is constant on arbitrarily long intervals.
3. For every n, $g(n + 1) \leq 2g(n)$.

However, the fourth conjecture is false—there are not infinitely many values of n for which $g(n+1) = 2g(n)$. On the contrary, Nicolas showed in [14] that

$$\lim_{n\to\infty} \frac{g(n+1)}{g(n)} = 1.$$

As for the fifth conjecture, it remains open. In [13], Nicolas conjectured that for each $k \geq 2$, there are infinitely many n with

$$g(n-1) = g(n) < g(n+1) < \cdots < g(n+k-1) = g(n+k).$$

In [4], Deléglise, Nicolas, and Zimmermann noted that there are only nine values of $n \leq 10^6$ with $k \geq 7$, and the current "record" value of k is $k = 20$, starting at $n = 35,464$.

Another active area of research for Landau's function involves the prime factorization of values of $g(n)$. In order to examine some of the patterns in these factors, we offer the following table, taken from [16].

n	$g(n)$	prime factorization
25	1260	$4 \cdot 9 \cdot 5 \cdot 7$
27	1540	$4 \cdot 5 \cdot 7 \cdot 11$
28	2310	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
29	2520	$8 \cdot 9 \cdot 5 \cdot 7$
30	4620	$4 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
32	5460	$4 \cdot 3 \cdot 5 \cdot 7 \cdot 13$
34	9240	$8 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
36	13,860	$4 \cdot 9 \cdot 5 \cdot 7 \cdot 11$
38	16,380	$4 \cdot 9 \cdot 5 \cdot 7 \cdot 13$
40	27,720	$8 \cdot 9 \cdot 5 \cdot 7 \cdot 11$
41	30,030	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
42	32,760	$8 \cdot 9 \cdot 5 \cdot 7 \cdot 13$
43	60,060	$4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
47	120,120	$8 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
49	180,180	$4 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
53	360,360	$8 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
57	471,240	$8 \cdot 9 \cdot 5 \cdot 7 \cdot 11 \cdot 17$

Given a prime p and a positive integer m, we let $\nu_p(m)$ denote the largest integer k such that p^k divides m. In addition, for each n, we denote by P_n the largest prime that divides $g(n)$. Nicolas showed in [15] that: for primes $p < q$ with $\alpha = \nu_p(g(n))$ and $\beta = \nu_q(g(n))$, $\beta \leq \alpha+1$; $\nu_{P_n}(g(n)) = 1$ unless $n = 4$; and

$$P_n \sim \log g(n) \sim \sqrt{n \log n}.$$

In [13], Nicolas also established that for any prime p, there is a positive integer n such that the largest prime factor of $g(n)$ is p. Thus for each prime p, we may define n_p to be the smallest input for which p divides $g(n_p)$. Based on the following table for small values of p, it is tempting to conjecture that n_p is increasing as a function of p.

p	2	3	5	7	11	13	17	...
n_p	2	3	8	14	27	32	57	...

However, n_p is not increasing as a function of p. In fact, as seen in [10], the smallest counterexample occurs when $n_{67} > n_{71}$:

p	...	61	67	71	73	...
n_p	...	429	519	510	586	...

2.4 Recent Progress

Research continues on refining the asymptotic result in Landau's main theorem. For example, as noted in [10], Massias obtained the following bounds in 1984:

- For $n \geq 2$, $\log g(n) \leq \sqrt{n \log n}\left(1 + \dfrac{\log\log n}{2\log n}\right)$.
- For $n \geq 906$, $\quad \log g(n) \geq \sqrt{n \log n}$.

He also showed that $\max\limits_{n>1} \left\{ \dfrac{\log g(n)}{\sqrt{n \log n}} \right\} = 1.05313\ldots$, with the maximum attained at $n = 1,319,166$. In 1989, Massias, Nicolas, and Robin established improved bounds [10]:

- For $n \geq 3$, $\log g(n) \leq \sqrt{n \log n}\left(1 + \dfrac{\log\log n - 0.975}{2\log n}\right)$.
- For $n \geq 810$, $\log g(n) \geq \sqrt{n \log n}\left(1 + \dfrac{\log\log n - 2}{2\log n}\right)$.

They also conjectured that for $n \geq 4$,

$$\frac{P_n}{\sqrt{n \log n}} \leq 1.265\ldots\ ,$$

with the maximum value occurring when $n = 215$. In 2012, Deléglise and Nicolas proved this conjecture [3].

For more research involving the largest prime P_n that divides $g(n)$, we note the 1995 result of Grantham [7]:

- For $n \geq 5$, $P_n \leq 1.328\sqrt{n \log n}$.

In 2012, Deléglise and Nicolas also established the following results [3]:

- For $n \geq 1755$, $\quad P_n \geq \sqrt{n \log n}$.
- There are infinitely many n with $P_n > \log g(n)$ and infinitely many n with $P_n < \log g(n)$.

On the computational side, in 2008, Deléglise, Nicolas, and Zimmermann proved the following result [4]:

Let $N = 2^{23}3^{15}5^{10}7^{8}11^{7}13^{6}17^{6}[19-31]^{5}[37-79]^{4}[83-389]^{3}[397-9623]^{2}[9629-192678817]$, where $[p-q]$ denotes the product of all primes between p and q, inclusive. Then the value of $g(10^{15})$ is:

$$\left(\frac{192678823 \cdot 192678853 \cdot 192678883 \cdot 192678917}{389 \cdot 9539 \cdot 9587 \cdot 9601 \cdot 9619 \cdot 9623 \cdot 192665881}\right) N.$$

Remaining computational challenges for Landau's function include finding the value of $g(10^n)$ for $n \geq 16$ and finding a value of $k > 20$ with

$$g(n) < g(n+1) < \cdots < g(n+k-1).$$

2.5 Final Word

Over one hundred years after Landau discovered his main result, his function continues to hold a certain fascination for mathematicians. With research still underway in both the theoretical and the computational realms, one may wonder whether this interest might wane at some point. On the other hand, as related in David Burton's *Elementary Number Theory* [1], Peter Barlow predicted in his own 1811 number theory text that the eighth perfect number $2^{30}(2^{31}-1)$ would be "the greatest that ever will be discovered; for as they are merely curious, without being useful, it is not likely that any person will ever attempt to find one beyond it." Perhaps Burton's response applies to the ongoing work on Landau's function as well:

"Barlow underestimated obstinate human curiosity."

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