Properties of Composition Operators

Hallie Kaiser, Katy O'Malley, Grace Weeks
Taylor University
Advised by: Dr. Derek Thompson

Binormal Property

Definition. A truncated composition operator, T, is *binormal* when $TT^*T^*T = T^*TTT^*$.

Theorem 1. *If T is a 3×3 binormal matrix, then the possible forms for which T is unitarily equivalent are:*

(i) $\begin{bmatrix} n & 0 & 0 \\ 0 & 1 & m \\ 0 & 0 & -1 \end{bmatrix}$ *where n is an arbitrary complex number and $m > 0$.*

(ii) *a weighted permutation.*

(iii) $\begin{bmatrix} 0 & g & -1 \\ 0 & 1 & g \\ 0 & 0 & 0 \end{bmatrix}$ *where $g > 0$.*

(iv) $\begin{bmatrix} 0 & 0 & C \\ u_{21} & u_{22} & 0 \\ {}_{31} & u_{33} & 0 \end{bmatrix}$ *where $C > 0$ and $\begin{bmatrix} u_{21} & u_{22} \\ u_{31} & u_{33} \end{bmatrix}$ is unitary.*

3×3

We set each form from Theorem 1 and T equal according to the trace conditions of Specht's Theorem. For each of the 4 cases, we are able to identify the only plausible variables for operator T to be binormal. With these results, we then investigated if they held true.

4×4

The methods for solving a 3×3 operator do not apply to 4×4 truncated composition operators. Therefore, we are unable to confirm that all solutions had been found. We found that setting either a, b, c, or d to zero and using the n-normal operator simplified the entries enough for us to solve for the variables.

$n \times n$

Looking through the patterns as an $n \times n$ operator expanded, we established a general solution for all $n \times n$ operators and proved it using algebra upon the non-truncated composition operator, C_φ, where $\varphi = a + bx$.

Binormal Conclusions

For a $\mathbf{3 \times 3}$ truncated composition operator, T, we found the binormal solutions to be $(a = 0, b = 0)$, $(a = 0, c = 0)$, $(b = -1, c = 0)$, $(a = 0, b = -1)$, and

$$\left(a = -\frac{3}{2c}, b = 1, c = \frac{1}{\sqrt{2}} \sqrt{-2 + \left(\frac{\left(175 + 3\sqrt{3891}\right)}{4} \right)^{\frac{1}{3}} - \frac{13}{\left(2\left(175 + 3\sqrt{3891}\right)\right)^{\frac{1}{3}}}} \right).$$

For a $\mathbf{4 \times 4}$ truncated composition operator, T, we found the binormal solutions $(a = 0, b = 0)$, $(b = -1, c = 0, d = 0)$, $(a = 0, b = -1 d = -c^2)$, and $(a = 0, c = 0, d = 0)$.

For a $\boldsymbol{n \times n}$ truncated composition operator, T, we found the only solution is when T is upper triangular and $b = -1$.

N-Normal Relationship

Definition. A truncated composition operator, T, is *n-normal* when $T^*T^n = T^nT^*$.

Each binormal solution that we found for both a 3×3 and 4×4 operator was also n-normal for some integer n. Because we were not able to find all solutions for the 4×4, this is only a conjecture as we could not test all solutions. Also, this conjecture is not biconditional - there are solutions that work for n-normal that are not binormal.

Background

Work on $\mathbb{R}^3$ can be extended to similar operations up to $\mathbb{R}^n$. It is a natural extension of the standard inner product on $\mathbb{R}^n$. The trick is identifying Taylor series as vectors, and normal vectors as coefficients of polynomials.

Diagonalization is an important concept in linear algebra because it simplifies many matrix computations. It is known that all normal matrices are diagonalizable and binormality is a more general form of normality.

A related concept is that matrices do not have a general commutative property like real numbers do. However, when two matrices do commute, they share many useful properties. For instance, normality and binormality are based off specific commutative relationships.

Truncated Composition Operator

Define a composition matrix, T, as:

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2 & x3 & \cdots \end{array} \\ \begin{bmatrix} 1 & a & a^2 & a^3 & \cdots \\ 0 & b & 2ab & 3a^2b & \cdots \\ 0 & c & 2ac + b^2 & 3a^2c + 3ab^2 & \cdots \\ 0 & d & 2ad + 2bc & 3a^2d + 6abc + b^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}.$$

Helpful Theorems

Lemma 1. *If the matrix A is binormal and unitarily equivalent to the matrix B, then B is also binormal.*

Theorem 2. ***Specht's Theorem:*** *The 3×3 matrices A and B are unitarily equivalent if and only if the following trace conditions are met:*

$\mathrm{Tr}(A) = \mathrm{Tr}(B)$,
$\mathrm{Tr}(A^2) = \mathrm{Tr}(B^2)$,
$\mathrm{Tr}(A^3) = \mathrm{Tr}(B^3)$,
$\mathrm{Tr}(AA^*) = \mathrm{Tr}(BB^*)$,
$\mathrm{Tr}(A^2A^*) = \mathrm{Tr}(B^2B^*)$,
$\mathrm{Tr}(A^2(A^*)^2) = \mathrm{Tr}(B^2(B^*)^2)$,
$\mathrm{Tr}(A^2(A^*)^2AA^*) = \mathrm{Tr}(B^2(B^*)^2BB^*)$

Questions

- We were unable to find final solutions in the unitary case (fourth case in Campbell's theorem). This would confirm all solutions for the 3×3 binormal composition matrix.
- We started work with the 4×4 binormal solutions, but could not confirm we had all answers. Are there any non-zero binormal solutions?
- Similarly, can we prove that all 4×4 binormal operators are also n-normal? what about $n \times n$?
- Are there any general commutative cases that will work for any $n \times n$ operator?

References

[1] Stephen L. Campbell, *Linear operator for which tt^* and t^*t commute (II)*, Pacific Journal of Mathematics **53** (1974), no. 2, 360.

[2] Caleb Holleman, Thaddeus McClatchey, and Derek Thompson, *Binormal, complex symmetric operators*, 2017.

[3] K. S. Sibirskiĭ, *Algebraic invariants of differential equations and matrices*, 1976.

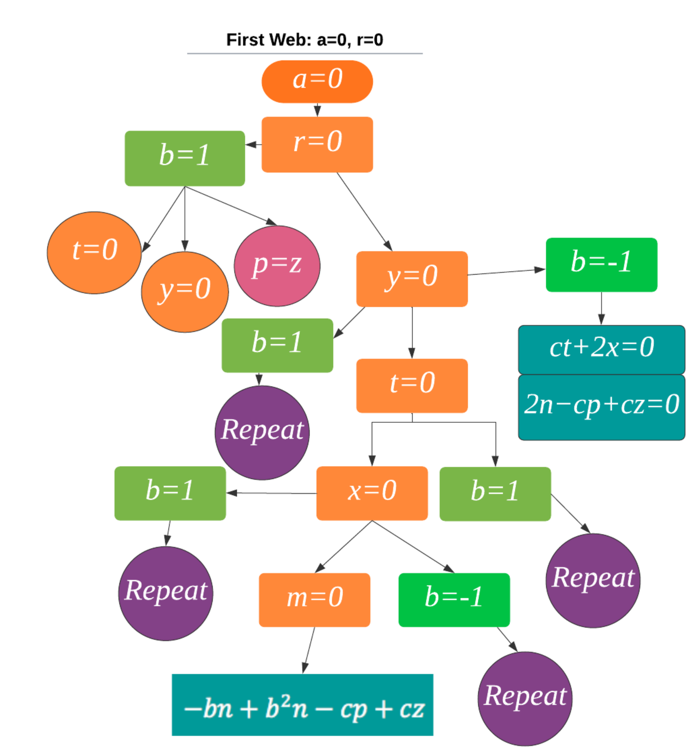
Commutative property

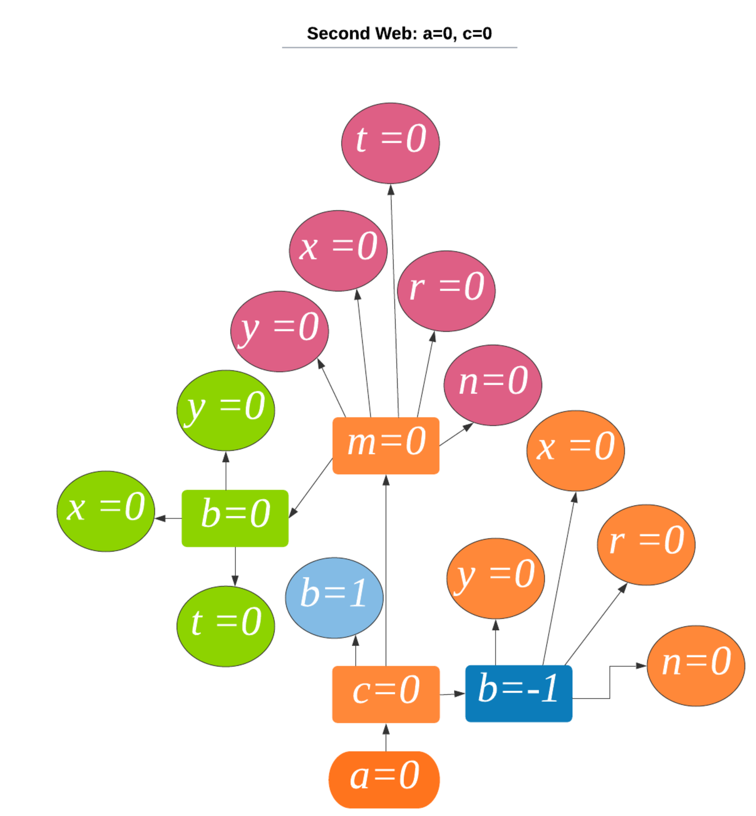
Define a generic 3×3 matrix, D, as:

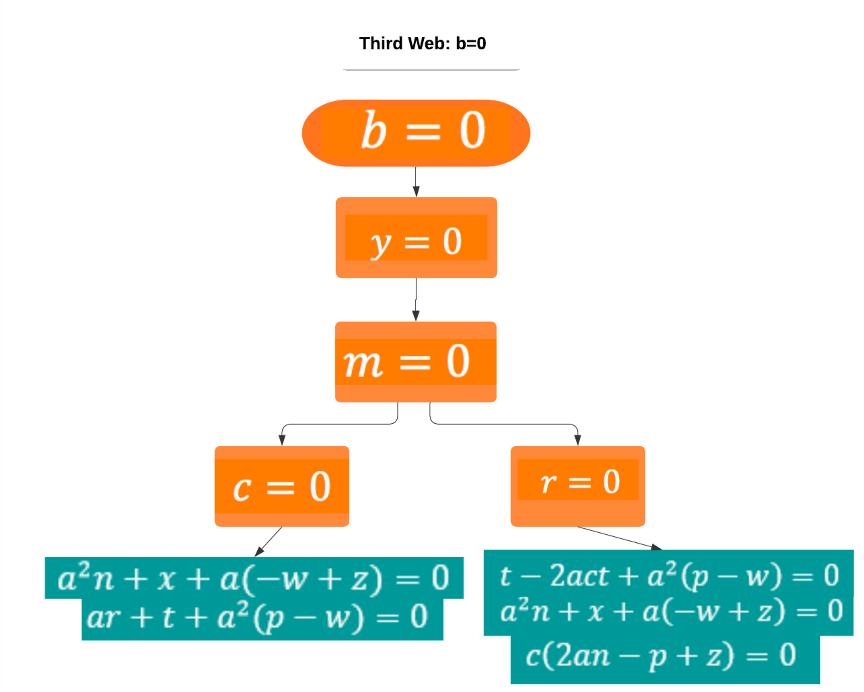
$$\begin{bmatrix} w & x & t \\ y & z & r \\ m & n & p \end{bmatrix}.$$

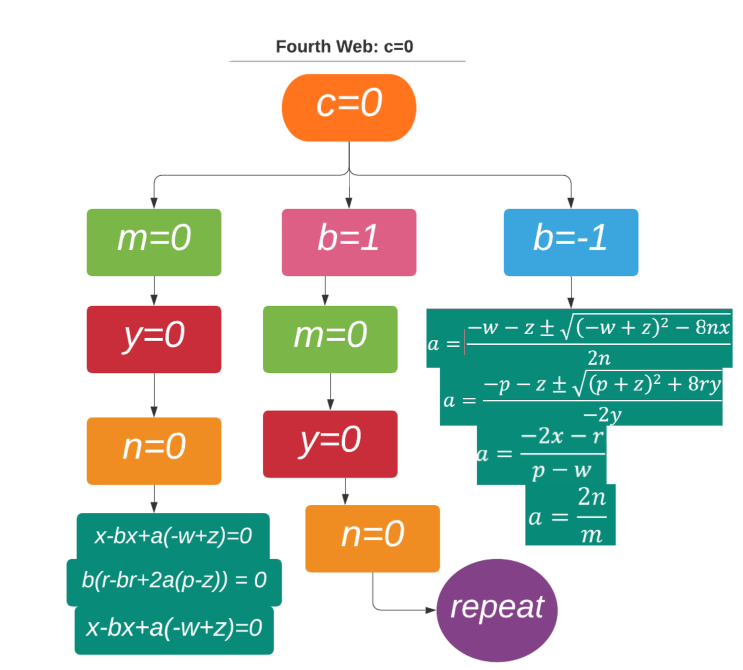
Definition. An operator, T, is commutative if, for a generic matrix D, $DT = TD$.

By definition, in order for T to be commutative, it must be true that $TD - DT = 0$. Thus, to find our solutions, we took the matrix $TD - DT$ and set all 9 entries equal to zero. We were then able to divide our proof into four different cases where $a = 0$, $b = 0$, $c = 0$, and $a, b, c \neq 0$. For each case, we would set our starting variable and then look at the reduced matrix. We would then look for equations in which there were simple solutions. There was often more than one such solution, so we would evaluate one of the solutions, look at the reduced matrix, and once again solve for simple equations and then go back and evaluate the next solution, etc. When it came down to equations that no longer had simple solutions, we just ended there, because whenever those equations were simultaneously equal to zero, then the matrices commuted. Each of the webs below outline the proof for a certain case.









For our last web, because $a, b, c \neq 0$, it proved true that y and m must be equal to zero.