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Normality Properties of Composition Matricies

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NORMALITY PROPERTIES OF COMPOSITION OPERATORS

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1. Introduction.

Composition operators are defined by $C_\varphi g = g \circ \varphi$ for some symbol φ in any space of analytic functions. These operators are most commonly studied on H^2, the Hilbert space of analytical functions on the complex unit disk with square-summable Taylor series. To make these operators work in finite dimesnions, we consider them on real polynomial subspaces of H^2, truncating as needed. [3]

As an example, let $\varphi(x) = \frac{1}{2-x}$ and consider C_φ on the real functions of degree two or less, with basis $1, x, x^2$. We will construct the matrix by applying the action of C_φ to each basis element, then removing any terms with degree 3 or higher:

$$C_\varphi 1 = 1 \circ \varphi = 1 \tag{1.1}$$

$$C_\varphi x = x \circ \varphi = \varphi = \frac{1}{2-x} = \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 \tag{1.2}$$

$$C_\varphi x^2 = x^2 \circ \varphi = \varphi^2 = \left(\frac{1}{2-x}\right)^2 = \frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 \tag{1.3}$$

Note that in the second and third calculations, the Taylor series for φ and φ^2 continue, but we do not want terms of degree above 2. Our matrix becomes:

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{bmatrix} 1 & 1/2 & 1/4 \\ 0 & 1/4 & 1/4 \\ 0 & 1/8 & 3/16 \end{bmatrix} \end{array}$$

More generally, if $\varphi(x) = a + bx + cx^2 + \dots$, then we have

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$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & x & x^2 & \cdots \end{array} \\
\begin{array}{c} 1 \\ x \\ x^2 \\ \vdots \end{array} & \begin{bmatrix} 1 & a & a^2 & \cdots \\ 0 & b & 2ab & \cdots \\ 0 & c & b^2+2ac & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\end{array}
$$

where this is process is doable for any truncation to a finite dimension n.

In linear algebra, an important concept is diagonalization because it simplifies many matrix computations — matrix multiplication, finding eigenspaces, projection, invertibility, etc. By definition all normal matrices are diagonalizable, and in this paper we explore the closely related concept of binormality — as it is a more generalized property of normality — and the conditions required for a composition matrix to be binormal. Another significant property is that matrices do not have the general commutative property that real numbers do. However, when two matrices are commutative, there are many useful properties between them; for example, binormality and normality are both defined by their ability to commute with their transpose. In this paper we also explore the conditions required to allow the composition matrix and a general matrix to commute. Furthermore, normal and binormal matrices commute with their transposes, and we also explore the conditions required for which a general composition matrix commutes with any matrix.

1.1. Preliminary.

Throughout our definitions, T will represent the truncated composition operator composed of all real numbers, T* will represent the transpose. We will begin with some important definitions:

Definition. A 3×3 truncated composition matrix, T, *commutes* with the generic matrix D when $TD = DT$.

Definition. A matrix, T, is *normal* if T commutes with T^*.

Definition. A matrix, T, is *binormal* if $TT^*T^*T = T^*TTT^*$.

Definition. A matrix, T, is *n-normal* if $T^nT^* = T^*T^n$ for some integer n. Equivalently, T is n-normal if T^n is normal. In the case of n=2, T is also binormal by [2, Theorem 1].

Definition A matrix, U is *unitary* when $U^{-1} = U^*$.

Definition Two matrices A and B are *unitarily equivalent* if there exists a unitary matrix U such that $B = U^*AU$.

The following lemma allows us to apply theorems that will ultimately lead to all binormal truncated composition matrices.

LEMMA 1.1. *If the matrix A is binormal and unitarily equivalent to the matrix B, then B is also binor-*

mal.

Proof. Let A be a matrix that is unitarily equivalent to the matrix B. By definition, $A = U^{-1}BU$, where U is a unitary matrix such that $U^{-1} = U^*$, thus $U^{(-1)^*} = U^{*(}-1)$. Let A also be binormal. By definition, $AA^*A^*A = A^*AAA^*$. When we substitute $A = U^{-1}BU$, we obtain the following. Remember as we simplify that $(U^{-1})^* = U$ because it is equivalent to $(U^*)^*$.

$$U^{-1}BU(U^{-1}BU)^*(U^{-1}BU)^*U^{-1}BU = (U^{-1}BU)^*U^{-1}BUU^{-1}BU(U^{-1}BU)^*$$
$$U^*BUU^*B^*UU^*B^*UU^*BU = U^*B^*UU^*BUU^*BUU^*B^*U$$
$$U^*BB^*B^*BU = U^*B^*BBB^*U.$$

When we multiply both sides by U on the left and then U^* on the right,

$$U(U^*BB^*B^*BU)U^* = U(U^*B^*BBB^*U)U^*$$
$$BB^*B^*B = B^*BBB^*.$$

By definition, B is also binormal. This completes the proof. ∎

Below is a theorem developed by Stephen Campbell who initiated studies on binormality in 1972. First, a quick definition: a *weighted permutation* matrix has exactly one non-zero entry in each row or column, and 0s elsewhere. If all the non-zero entries are 1s, then the matrix is simply a permutation matrix.

THEOREM 1.2. *[1] If T is a 3×3 binormal matrix, then the possible forms for which T is unitarily equivalent are:*

1. $W = \begin{bmatrix} n & 0 & 0 \\ 0 & 1 & m \\ 0 & 0 & -1 \end{bmatrix}$ *where n is an arbitrary complex number and $m > 0$.*
2. $P =$ *a weighted permutation.*
3. $V = \begin{bmatrix} 0 & g & -1 \\ 0 & 1 & g \\ 0 & 0 & 0 \end{bmatrix}$ *where $g > 0$.*
4. $X = \begin{bmatrix} 0 & 0 & C \\ u_{21} & u_{22} & 0 \\ u_{31} & u_{33} & 0 \end{bmatrix}$ *where $C > 0$ and $\begin{bmatrix} u_{21} & u_{22} \\ u_{31} & u_{33} \end{bmatrix}$ is unitary.*

It's worth noting that this theorem has not been extended to 4×4 matrices and is only applicable to 3×3 matrices.

THEOREM 1.3. ***Specht's Theorem:*** *[4] The 3×3 matrices A and B are unitarily equivalent if and only if the following trace conditions are met:*

$$\mathrm{Tr}(A) = \mathrm{Tr}(B) \tag{1.4}$$

$$\mathrm{Tr}(A^2) = \mathrm{Tr}(B^2) \tag{1.5}$$

$$\mathrm{Tr}(A^3) = \mathrm{Tr}(B^3) \tag{1.6}$$

$$\mathrm{Tr}(AA^*) = \mathrm{Tr}(BB^*) \tag{1.7}$$

$$\mathrm{Tr}(A^2A^*) = \mathrm{Tr}(B^2B^*) \tag{1.8}$$

$$\mathrm{Tr}(A^2(A^*)^2) = \mathrm{Tr}(B^2(B^*)^2) \tag{1.9}$$

$$\mathrm{Tr}(A^2(A^*)^2AA^*) = \mathrm{Tr}(B^2(B^*)^2BB^*) \tag{1.10}$$

We are able to set the equations from the above trace conditions equal, where B is one of the conditions from 1.2 and A is our matrix T, in order to find conditions on the entries of T that result in T being binormal.

2. Normal Composition Matrices.

Because binormality is a more generalized property of normality, any normal matrix is also binormal. Thus, we check for normal solutions. The following theorems identifies this solution for the 3×3 and then the 4×4.

Theorem 2.1. *A 3×3 truncated composition matrix is normal if and only if $a = 0$ and $c = 0$.*

Proof. Recall our 3×3 truncated composition matrix:

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{bmatrix} 1 & a & a^2 \\ 0 & b & 2ab \\ 0 & c & b^2 + 2ac \end{bmatrix} \end{array}$$

First, let our 3×3 truncated composition matrix, T, be normal. By definition, $TT^* - T^*T = 0$. Below is the matrix, $TT^* - T^*T = 0$.

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{bmatrix} a^2 + a^4 & -a + ab + 2a^3b & -a^2 + a^2(2ac + b^2) + ac \\ -a + ab + 2a^3b & -a^2 + 4a^2b^2 - c^2 & A \\ -a^2 + a^2(2ac + b^2) + ac & B & -a^4 - 4a^2b^2 + c^2 \end{bmatrix} \end{array}$$

where $A = -a^3 - 2ab^2 + 2ab(2ac + b^2) + bc - (2ac + b^2)c$ and $B = -a^3 - 2ab^2 + 2ab(2ac + b^2) + bc - (2ac + b^2)c$.

Therefore, every entry in the matrix $TT^* - T^*T$ must be equal to 0. When we calculate entry t_{11} of $TT^* - T^*T$, we observe that $a^2 + a^4 = a^2(1 + a^2) = 0$ only when $a = 0$. We regard t_{33}, $a^4 - 4a^2b^2 + c^2$, and plug in $a = 0$, which then simplifies the entry to $c^2 = 0$. Therefore, $c = 0$ as well. From further calculations we know that this solution of $(a = 0, c = 0)$ also works for all other entries of this matrix. Thus, this is the only solution when the matrix is normal.

Next, let $a = 0$ and $c = 0$.

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^2 \end{bmatrix} \end{array}$$

Thus, T becomes a symmetric matrix. By definition of symmetric matrices, T commutes with T^*. Therefore, T is normal and this completes the proof. □

THEOREM 2.2. *A* 4×4 *truncated composition matrix is normal if and only if* $a = 0$, $c = 0$, *and* $d = 0$.

Proof. Recall our 4×4 truncated composition operator:

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & b & 2ab & 3a^2b \\ 0 & c & 2ac+b^2 & 3a^2c+3ab^2 \\ 0 & d & 2ad+2bc & 3a^2d+6abc+b^3 \end{bmatrix} \end{array}.$$

First, let our 4×4 truncated composition matrix be normal. By definition, $TT^* - T^*T = 0$, Therefore, every entry in the matrix, $TT^* - T^*T$ must be equal to zero. Entry t_{11} is $a^2 + a^4 + a^6 = 0$ which means that a must equal zero. Setting $a = 0$ and calculating the new matrix, reveals entry t_{22} as $-c^2 - d^2 = 0$. Because both terms are squared, the only way for this to equal zero is if both c and d equal zero. When we set $a, c, d = 0$, all entries are zero.

Next, let $a = 0$, $c = 0$, and $d = 0$.

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0+b^2 & 0 \\ 0 & 0 & 0 & b^3 \end{bmatrix} \end{array}.$$

Thus, T becomes a symmetric matrix. By definition of symmetric matrices, T commutes with T^*. Therefore, T is normal and this completes the proof. □

Based on the similarity of the two proofs above, we speculate that the pattern continues up to $n \times n$.

CONJECTURE 2.1. *For any* $n \times n$ *composition matrix, it is normal if* b *is a free variable and all other variables are zero.*

When all variables are set to zero except for b, this creates a diagonal matrix. Thus, the matrix is also symmetric, which is by definition normal.

3. Binormal Composition Matrices.

In Section 3.1 we find 3×3 truncated composition matrices that are binormal, and consider their relationship to the n-normal property. In Section 3.2, we will explore 4×4 truncated composition matrices that are binormal and their relationship to the n-normal property. Finally, in section 3.3, we examine a $n \times n$ truncated composition matrix which is binormal.

3.1. 3×3 Matrix.

We know that there are more composition matrices that are binormal than the normal, because it is more generalized. Thus, because of Lemma 1.1, we are able to use theorem 1.2 and theorem 1.3 to find those matrices. Below are four proofs that cover the four cases laid out by theorem 1.2.

3.1.1. Campbell's Theorem Case 1:.

THEOREM 3.1. *Suppose a* 3×3 *truncated composition matrix,* T*, is unitarily equivalent to the matrix* W *(see Theorem 1.2), given by*

$$\text{W} = \begin{bmatrix} n & 0 & 0 \\ 0 & 1 & m \\ 0 & 0 & -1 \end{bmatrix}$$

FIGURE 3.1. *Matrix representation of* W

where n *is an arbitrary complex number and* $m > 0$*. Then, the only solutions that will produce a binormal matrix are* $(a = 0, b = -1)$ *or* $(c = 0, b = -1)$*.*

Proof. Let the truncated composition matrix, T, be unitarily equivalent to the matrix W (see figure 3.1 above). Theorem 1.3 (Specht's Theorem) states the seven trace conditions that must be met for two matrices to be unitarily equivalent. We will start with the 1.4:

$$1 + b + b^2 + 2ac = n.$$

Thus, from now on we will let $n = 1 + b + b^2 + 2ac$.

The next condition is 1.5:

$$1 + b^2 + b^4 + 4abc + 4ab^2c + 4a^2c^2 = 3 + 2b + 3b^2 + 2b^3 + b^4 + 4ac + 4abc + 4ab^2c + 4a^2c^2 \tag{3.1}$$
$$1 + b^2 = 3 + 2b + 3b^2 + 2b^3 + 4ac \tag{3.2}$$
$$0 = 2 + 2b + 2b^2 + 2b^3 + 4ac \tag{3.3}$$
$$2ac = -1 - b - b^2 - b^3. \tag{3.4}$$

The next condition is 1.6:

$$1+b(b^2+2abc)+c(2ab^2+2ab(b^2+2ac))+2ab(bc+c(b^2+2ac))+(b^2+2ac)(2abc+(b^2+2ac)^2) = (1+b+b^2+2ac)^3.$$

When we expand and simplify, we observe

$$b^3 + 6ab^2c + 6ab^3c = 3b + 6b^2 + 7b^3 + 6b^4 + 3b^5 + 6ac + 12abc + 18ab^2c + 12ab^3c + 12a^2c^2$$
$$0 = b + 2b^2 + 2b^3 + 2b^4 + b^5 + 2ac + 4abc + 4ab^2c + 2ab^3c + 4a^2c^2.$$

By rewriting our equation in standard quadratic form

$$4a^2c^2 - c(2a + 4ab + 4ab^2 + 2ab^3) + b + 2b^2 + 2b^3 + 2b^4 + b^5 = 0$$

and using the quadratic formula to solve:

which produces,

$$b + b^2 = -b - b^2$$
$$2(b + b^2) = 0.$$

Thus, $b = -1$ or $b = 0$.

Referring back to Equation 3.4 and plug in $b = -1$:

$$2ac = -1 - b - b^2 - b^3$$
$$2ac = 0.$$

Therefore, either a or c must equal zero. These solutions, when checked, create a binormal matrix T. We know that these must be the only plausible solutions for the other four conditions as well.

Then, we plug in $b = 0$

$$2ac = -1 - b - b^2 - b^3$$
$$2ac = -1$$
$$a = \frac{-1}{2c}.$$

When the solution $(a = \frac{-1}{2c}, b = 0)$ is checked, it is not binormal. Therefore, there must be at least one of the seven trace conditions that is not true and the matrices are not unitarily equivalent for $(a = \frac{-1}{2c}, b = 0)$.

Thus, the only answers that work for all three conditions here are when $(a = 0, b = -1)$ or $(c = 0, b = -1)$. This completes the proof. ◻

3.1.2. Campbell's Theorem Case 2:.

THEOREM 3.2. *If a 3×3 truncated composition matrix, T, is unitarily equivalent to the matrix V (see Theorem 1.2), given by,*

$$\mathrm{V} = \begin{bmatrix} 0 & g & -1 \\ 0 & 1 & g \\ 0 & 0 & 0 \end{bmatrix}$$

FIGURE 3.2. *Matrix representation of V*

where $g > 0$. Then, the only binormal solution is $(a = 0, b = 0)$.

Proof. Let the truncated composition matrix, T, be unitarily equivalent to the matrix V (see figure 3.2 above). We again use Theorem 1.3 (Specht's Theorem) to solve for the variables of T.

The first condition is 1.4:

$$1+b+b^2+2ac=1 \tag{3.5}$$
$$b+b^2+2ac=0 \tag{3.6}$$
$$b+b^2=-2ac. \tag{3.7}$$

The next condition is 1.5:

$$1+b^2+4abc+(b^2+2ac)^2=1$$
$$b^2+b^4+4abc+4ab^2c+4a^2c^2=0$$

By rewriting our equation in the standard quadratic form:

$$4a^2c^2+c(4ab+4ab^2)+b^2+b^4=0,$$

We use the quadratic formula which reduces down to,

$$\sqrt{2}b^{\frac{3}{2}}-b-b^2=-\sqrt{2}b^{\frac{3}{2}}-b-b^2$$
$$\sqrt{2}b^{\frac{3}{2}}=-\sqrt{2}b^{\frac{3}{2}}$$
$$-2(\sqrt{2}b^{\frac{3}{2}})=0.$$

Thus, $b=0$. When we go back to Equation 3.7 and plug in $b=0$,

$$-b-b^2=2ac$$
$$0=2ac.$$

Either a or c must be zero. Thus, the only answers that allow for both statements to be true are when $(a=0,b=0)$ or $(c=0,b=0)$. When we check the solution,$(c=0,b=0)$, it is not binormal; there must be at least one of the seven trace conditions that is not true and the matrices are not unitarily equivalent, when $(b=0,c=0)$. However, when we check $(a=0,b=0)$, T is binormal. Thus, we know that this must be the only plausible solution for the other five conditions as well and the matrices are unitarily equivalent when $(a=0,b=0)$. This completes the proof. ∎

3.1.3. Campbell's Theorem Case 3.

THEOREM 3.3. *Suppose a 3×3 composition matrix is unitarily equivalent to a weighted permutation, P. Then the solutions are $(a=0,c=0)$,$(a=0,b=-1)$, and $(a=\frac{-3}{2c}, b=1,$*
$c=\frac{1}{\sqrt{2}}\sqrt{-2+\left(\frac{(175+3\sqrt{3891})}{4}\right)^{\frac{1}{3}}-\frac{13}{(2(175+3\sqrt{3891}))^{\frac{1}{3}}}})$

Proof. If P is a weighted permutation in M_3, then either P, P^2, or P^3 is diagonal. A matrix is unitarily equivalent to a diagonal matrix if and only if the matrix is normal, so if $T=\widetilde{C_\varphi}$ is unitarily equivalent to P, then either T, T^2, or T^3 is normal.

P **is diagonal.** Here T is normal. T is only normal when $a = c = 0$. (We proved this above in theorem 2.1)

P^2 **is diagonal.** Here P is of the form

$$\begin{bmatrix} 0 & 0 & p \\ 0 & q & 0 \\ r & 0 & 0 \end{bmatrix}$$

where $p, q, r \in \mathbb{C}$ (p or r could be on the diagonal instead, but we will use q without loss of generality). Here P has only one element on the diagonal. By Specht's Theorem, $q = 1 + b + b^2 + 2ac$. However, q is also an eigenvalue of P, with the other two eigenvalues being $\pm\sqrt{pr}$. Note that 1 is an eigenvalue for T. Suppose momentarily that either $\sqrt{pr}$ or $-\sqrt{pr}$ is 1, so that $pr = 1$. Then $\operatorname{Tr} P^2 = q^2 + 2pr = (1 + b + b^2 + 2ac)^2 + 2, \operatorname{Tr}(T^2) = 1 + b^2 + 4abc + (b^2 + 2ac)^2$, and by Specht's Theorem,

$$\frac{1}{2}(\operatorname{Tr}(P^2) - \operatorname{Tr}(M^2)) = 1 + b + b^2 + b^3 + 2ac = 0. \tag{3.8}$$

From there, we can set $a = -\frac{1+b+b^2+b^3}{2c}$. Since T is assumed to be unitarily equivalent to P, T^2 is unitarily equivalent to P^2, which is diagonal, so T^2 is normal. Therefore $(a_{jk}) = TTT^*T^* - T^*T^*TT = 0$. After substituting for a, then we have $(a_{11}) = \frac{(b-1)^2(b+1)^6(b^2+1)^2(b^2+2b^4+b^6+c^2)}{16c^4}$. Since $b, c \in \mathbb{R}$, $b^2 + 1 \neq 0$ and $b^2 + 2b^4 + b^6 + c^2 = 0$ unless $b = c = 0$, which we proved. Otherwise, the only solution is that $b = \pm 1$. If $b = -1$, then $a = 0$, which we have covered. If $b = 1$, then $(a_2 2) = \frac{64-4c^4}{c^2} = 0$, so $c = \pm 2$. If $c = \pm 2$, then $(a_3 2) = \pm 64 \neq 0$, so this is impossible. Therefore, no 3×3 composition matrix is unitarily equivalent to a permutation of order two.

P^3 **is diagonal.** Here, P takes the form

$$\begin{bmatrix} 0 & 0 & p \\ q & 0 & 0 \\ 0 & r & 0 \end{bmatrix}$$

for some $p, q, r \in \mathbb{C}$. In this case, P has 0 elements on the diagonal. By condition 1.4 of Specht's Theorem: $1 + b + b^2 + 2ac = 0$, so $a = \frac{-1-b-b^2}{2c}$. Continuing with Specht's Theorem, P^2 also has 0 elements on the diagonal, and now after substituting for a, $\operatorname{Tr}(T^2) = 2 - 2b^3 = 0$, so $b = 1$, and $a = -\frac{3}{2c}$.

Continuing with Specht's Theorem, we have $\operatorname{Tr}(P^2P^*) = 0$, while after substitution for a and b, we have $\operatorname{Tr}(M^2M^*) = -3 + \frac{81}{16c^4} - \frac{63}{8c^2} - c^2 = 0$. Simplifying this equation yields $16c^6 + 48c^4 + 126c^2 - 81 = 0$, which is a cubic in c^2. The cubic has only real solution, resulting in
$c = \frac{1}{\sqrt{2}}\sqrt{-2 + \left(\frac{(175+3\sqrt{3891})}{4}\right)^{\frac{1}{3}} - \frac{13}{\left(2\left(175+3\sqrt{3891}\right)\right)^{\frac{1}{3}}}}$.
Thus, T is binormal. This completes the proof. ∎

3.1.4. Campbell's Theorem Case 4.

In the last case, we have only a partial result.

Here we suppose X is of the form

$$X=\begin{bmatrix} 0 & 0 & C \\ A & B & 0 \\ -e^{i\theta}\overline{B} & e^{i\theta}\overline{A} & 0 \end{bmatrix}$$

where $C > 0$ and the bottom-left 2×2 block is the general form of a 2×2 unitary matrix over $\mathbb{C}$, with $|A|^2 + |B|^2 = 1$. We are able to narrow down the form of P, but not identify which composition matrices may be unitarily equivalent to such P.

By Specht's Theorem, $B = 1 + b + b^2 + 2ac$, so we know B is real.

Then $\text{Tr}(P^3) = B^3 + 3e^{i\theta}C|A|^2$, and $\text{Tr}(T^3)$ is real. Since B^3, C, and $|A|^2$ are real, this means that $e^{i\theta}$ is real and equal to ± 1. Continuing with condition 1.7 of Specht's Theorem, we have $\text{Tr}(PP^*) = C^2 + 2 = \text{Tr}(TT^*) = 1 + a^2 + a^4 + b^2 + 4a^2b^2 + b^4 + 4ab^2c + c^2 + 4a^2c^2$, and since $C > 0$, we have $C = \sqrt{a^2 + a^4 + b^2 + 4a^2b^2 + b^4 + 4ab^2c + c^2 + 4a^2c^2 - 1}$.

3.1.5. N-Normality.

After we had found 3×3 truncated composition matrices that were binormal, we attempted to find the 4×4 truncated composition matrices that were binormal, but found that the methods we had been using were not applicable. Thus, we considered the relationship between binormal and n-normal. We started by looking at the 3×3 matrix's relationship in order to see if it could apply to the 4×4 as well. Because our last case above was only a partial result, we are unable to be sure that we have every matrix and cannot be sure that every binormal will also be n-normal. Below is our conjecture based on the relationship we found.

CONJECTURE 3.1. *If a 3×3 truncated composition matrix, T, is binormal, then T is also n-normal for some integer n.*

Let our 3×3 truncated composition matrix, T, be binormal. We have proved above that the only binormal solutions to T are $(a = 0, b = 0), (a = 0, b = -1), (c = 0, b = -1)$, and $(a = -\frac{3}{2c}, b = 1, c = \frac{1}{\sqrt{2}}\sqrt{-2 + \left(\frac{(175+3\sqrt{3891})}{4}\right)^{\frac{1}{3}} - \frac{13}{\left(2\left(175+3\sqrt{3891}\right)\right)^{\frac{1}{3}}}})$. When we enter our binormal solutions into the n-normal equation:

$$\boldsymbol{a = 0, b = 0}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{a = 0, b = -1}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & c\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & c\\ 0 & 0 & 1\end{bmatrix}$$

$$\boldsymbol{b = -1, c = 0}$$

$$\begin{bmatrix}1 & 0 & 0\\ a & -1 & 0\\ a^2 & -2a & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ a & -1 & 0\\ a^2 & -2a & 1\end{bmatrix}$$

$$\boldsymbol{a = -\frac{3}{2c}, b = 1}$$

$$\begin{bmatrix}1 & 0 & 0\\ -3/2c & 1 & c\\ 9/4c^2 & -3/c & -2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -3/2c & 1 & c\\ 9/4c^2 & -3/c & -2\end{bmatrix}.$$

All solutions are for $n = 2$, except the solution $(a = \frac{-3}{2c}, b = 1)$ will create an n-normal for $n = 3$ with any value for c. Thus, our binormal solution $(a = -\frac{3}{2c}, b = 1,$
$c = \frac{1}{\sqrt{2}}\sqrt{-2 + \left(\frac{(175+3\sqrt{3891})}{4}\right)^{\frac{1}{3}} - \frac{13}{(2(175+3\sqrt{3891}))^{\frac{1}{3}}}}$) will be n-normal.

Because for each binormal solution, the statement $T^nT^* = T^*T^n$ is true, we have proven that each binormal solution is also n-normal.

It is worth noting that this relationship is not biconditional. For example, the final solution in Theorem 3.1 needs c to be a specific value in order to be binormal but not to be n-normal.

3.2. 4×4 Matrix.

The following is the 4×4 truncated composition matrix:

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & b & 2ab & 3a^2b \\ 0 & c & 2ac + b^2 & 3a^2c + 3ab^2 \\ 0 & d & 2ad + 2bc & 3a^2d + 6abc + b^3 \end{bmatrix} \end{array}.$$

Because the 4×4 binormal matrices generates much more complicated entries and Theorem 1.2 only applies to 3×3 matrices, we were not able to confirm that we found all possible matrices. We found that setting either a, b, c, or d to zero and using the n-normal matrix simplified the entries enough for us to solve for the variables. The theorem below proves only the matrices that we were able to find.

THEOREM 3.4. *A 4×4 truncated composition matrix, T, is binormal when* $(a = 0, b = 0)$, $(a = 0, c = 0, d = 0)$, $(a = 0, b = -1, d = -c^2)$, *or* $(b = -1, c = 0, d = 0)$.

Proof. Let T represent a 4×4 truncated composition matrix. It has been proven [5] that if T^2 is normal, then T is binormal. Thus, let T be n-normal for $n=2$, and therefore T is binormal. In order for an matrix to be n-normal, it must be true that $T^nT^* - T^*T^n = 0$. We will have four cases, $a=0$, $b=0$, $c=0$, and $d=0$. We will plug each of the four cases into the n-normal matrix, $T^2T^* - T^*T^2$, to solve for the binormal solutions.

First, let $a=0$, which reduces t_{23} and t_{24} down to,

$$\begin{aligned} -\boldsymbol{b^2}(1+b)\boldsymbol{c}(-1+b+2bd) &= 0 \\ -\boldsymbol{b^2}(-1+b^4)\boldsymbol{d} &= 0. \end{aligned}$$

Between these two entries, the common solutions are $(a=0, b=0)$, $(a=0, c=0, d=0)$, and $(a=0, b=-1)$. By direct calculation we can confirm that the solutions, $(a=0, b=0)$ and $(a=0, c=0, d=0)$, is true for every other entry in the matrix. However, when we plug in $(a=0, b=-1)$, the matrix reduces down to these four entries,

$$\begin{aligned} 2d(\boldsymbol{c^2+d}) &= 0 \\ -4c(\boldsymbol{c^2+d}) &= 0 \\ -2c(\boldsymbol{c^2+d}) &= 0 \\ -2d(\boldsymbol{c^2+d}) &= 0. \end{aligned}$$

From here we can see the solution of $d=-c^2$ works for all of the above entries. Thus, a final solution is $(a=0, b=-1, d=-c^2)$. We can also see the solution of $(a=0, c=0, d=0)$ which we have already prove above.

Next, let $b=0$, which reduces t_{21},

$$-a = 0.$$

Thus, the only solution is $(a=0, b=0)$, which is proved above. Therefore, setting $b=0$ does not give any new solutions.

Next, we will look at the $c=0$ case, which reduces t_{14} and t_{22}

$$\begin{aligned} \boldsymbol{a}\boldsymbol{(}a^2(-1+b)\boldsymbol{(1+b)^3}(1+b+b^2)+\boldsymbol{d}(1+2a^2+b+4a^2b+9a^4b+2a^2b^2+9a^4b^2+6a^4b^3+a^2d+4a^4d+9a^6d)\boldsymbol{)} &= 0 \\ \boldsymbol{a^2(1+b)}(-1+4b^3+9a^2b^3+9a^2b^4)+\boldsymbol{d}(-a^4+12a^4b^2+27a^6b^2-bd-b^3d-3a^2d^2) &= 0. \end{aligned}$$

There are only two common solutions between these two terms. From here we can see the solution of $(a=0, c=0, d=0)$, which has already been proven above. We can also see the new solution of $(b=-1, c=0, d=0)$. Through direct calculation, we know that this is a real solution for the whole matrix.

Finally, we will look at the $d=0$ case, which reduces the entries t_{13} and t_{23},

$$\boldsymbol{a(1+b)^3}(-1+b+3a^2b^2)+\boldsymbol{c}(1+3a^4+b+2a^2b+9a^4b+4a^2b^2+18a^4b^2+2a^2b^3+21a^4b^3$$
$$+ac+4a^3c+9a^5c+4a^3bc+18a^5bc)=0$$
$$\boldsymbol{a(1+b)}(-a^2-a^2b-2b^3+2b^4+9a^2b^4+9a^2b^5)+\boldsymbol{c}(-2a^4-2a^4b+b^2+9a^4b^2+2a^2b^3+36a^4b^3$$
$$-b^4+6a^2b^4+63a^4b^4+8a^3bc+18a^5bc-4ab^2c+12a^3b^2c+54a^5b^2c-6ab^3c-4a^2c^2-6a^2bc^2)=0.$$

From here, we can see the solutions of $(a=0, c=0, d=0)$ and $(b=-1, c=0, d=0)$, both of which have been proven above. There are no other common solutions between these two terms. Thus, we can conclude that there are no new solutions when $d=0$. Therefore, a 4×4 composition matrix, T is n-normal (for $n=2$) and therefore binormal when $(a=0, b=0)$, $(a=0, c=0, d=0)$, $(a=0, b=-1, d=-c^2)$, or $(b=-1, c=0, d=0)$. This completes the proof. □

We are unable to conclude that all 4×4 binormal matrices will be n-normal because we cannot be certain that we have every binormal matrix. However, since every 3×3 and every 4×4 binormal matrix we found is n-normal, we can speculate that all known and unknown 4×4 binormal matrices will be n-normal. Below is our evidence for this conjecture.

CONJECTURE 3.2. *If a 4×4 truncated composition matrix, T, is binormal, then T is also n-normal.*

Let T represent a binormal 4×4 truncated composition matrix. The binormal solutions for a 4×4 truncated composition matrix that we know are $(a=0, b=0)$,$(a=0, c=0, d=0)$, $(a=0, b=-1, d=-c^2)$, and $(b=-1, c=0, d=0)$. When we plug our binormal solutions into the n-normal equation for $n=2$:

$$\boldsymbol{a=0, b=0}$$

$$\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$\boldsymbol{a=0, c=0, d=0}$$

$$\begin{bmatrix}1&0&0&0\\0&b^3&0&0\\0&0&b^6&0\\0&0&0&b^9\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&b^3&0&0\\0&0&b^6&0\\0&0&0&b^9\end{bmatrix}$$

$$\boldsymbol{a=0, b=-1, d=-c^2}$$

$$\begin{bmatrix}1&0&0&0\\0&-1&c&-c^2\\0&0&1&-2c\\0&0&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&-1&c&-c^2\\0&0&1&-2c\\0&0&0&-1\end{bmatrix}$$

$$\boldsymbol{b=-1, c=0, d=0}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & -1 & 0 & 0 \\ a^2 & -2a & 1 & 0 \\ a^3 & -3a^2 & 3a & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & -1 & 0 & 0 \\ a^2 & -2a & 1 & 0 \\ a^3 & -3a^2 & 3a & -1 \end{bmatrix}.$$

Because for each binormal solution the statement $T^nT^* = T^*T^n$ is true, we know that each binormal solution that we have found for the 4×4 truncated composition matrix, T, is also n-normal for $n = 2$.

3.3. $n \times n$ Matrix.

As we solved for each binormal 3×3 truncated composition operator, we looked for patterns that stayed true for 4×4 matrices, 5×5 matrix, etc.. There was one matrix that we were able to generalize to all $n \times n$ matrices. This theorem discusses this solution and explains why we were able to generalize it.

THEOREM 3.5. *If any $n \times n$ composition matrix, T, is upper triangular with $b = -1$, then it will be binormal.*

Proof. Let $C_{\varphi(x)}$ represent our composition operator, where $\varphi(x) = a + bx$. When φ is a linear symbol, there is no truncation. Then, let T represent an $n \times n$ matrix representation of C_φ. Next, let $b = -1$, thus $\varphi(x) = a - x$, which creates an upper triangular matrix of the form:

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & \cdots \end{array} \\ \begin{bmatrix} 1 & a & a^2 & \cdots \\ 0 & -1 & -2a & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}.$$

When we square the composition operator, $C_{\varphi(x)} * C_{\varphi(x)} = C_{\varphi(x) \circ \varphi(x)}$ then we see:

$$\begin{aligned} \varphi(x) \circ \varphi(x) &= a - \varphi(x) \\ &= a - (a - x) \\ &= x. \end{aligned}$$

Thus, our function is x and our composition operator, C_x. Thus, the composition matrix T is:

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & \cdots \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}.$$

Therefore, $C_{\varphi(x)} * C_{\varphi(x)} = C_{\varphi(x) \circ \varphi(x)} = C_x = I$, where I represents the identity matrix. Because $C_{\varphi(x)} = T$, by substitution, $T^2 = I$. The definition of a binormal matrix is $TT^*T^*T - T^*TTT^* = 0$. Thus, we observe that:

$$TT^*T^*T = T(TT)^*T = T(I)^*T = TIT = TT = I$$

and

$$T^*TTT^* = T^*IT^* = T^*T^* = (TT)^* = (I)^* = I.$$

Thus,

$$I - I = 0.$$

Therefore, a $n \times n$ composition matrix, T, that is upper triangular with $b = -1$ is binormal. This completes the proof. ∎

4. Commutativity. Our truncated matrix T commutes with a generic matrix D when $TD - DT = 0$. Below is the generalized matrix D that we will be using.

$$D = \begin{bmatrix} w & x & t \\ y & z & r \\ m & n & p \end{bmatrix}$$

There are two obvious cases for communicative matrices. The first case would be when $T = D$. The second case would be when $D = 0$. For the purpose of finding the specific conditions that allow T to commute with D that do not align with the above cases, we will look at when the nine entries of the matrix $TD - DT$ are set equal to 0.

$$a(am + y) = 0 \tag{4.1}$$

$$2abm + (-1 + b)y = 0 \tag{4.2}$$

$$a^2n - ct + x - bx + a(-w + z) = 0 \tag{4.3}$$

$$t - b^2t + a^2(p - w) + a(r - 2(ct + bx)) = 0 \tag{4.4}$$

$$2abn - cr - ay = 0 \tag{4.5}$$

$$-(-1 + b)br - a^2y - 2a(cr + b(-p + z)) = 0 \tag{4.6}$$

$$(-1 + b^2 + 2ac)m + cy = 0 \tag{4.7}$$

$$-am - bn + b^2n + 2acn - cp + cz = 0 \tag{4.8}$$

$$-a^2m - 2abn + cr = 0. \tag{4.9}$$

In the proofs below, we will look at the conditions required when a, b, or c equal zero and when a, b, and c are all non-zero. At the end of Theorems 4.1, 4.2, and 4.3, there are webs which visually depict the path through the equations which led to solutions. Each web begins with either a, b or c equaling zero. The arrows leads to the next necessary variable solution with the number of the equation that the solution came from. Each cluster of circles at the end of a branch represents that branch's final answer.

4.1. $a = 0$ **Case.**

THEOREM 4.1. *If the matrix T commutes with other matrices when $a = 0$, then either r or c from matrix D are zero.*

Proof. Let $a = 0$. This reduces equations 4.5 and 4.9 to

$$-cr = 0 \tag{4.10}$$

$$cr = 0. \tag{4.11}$$

There are only two solutions to Equations 4.10 and 4.11: $r = 0$ or $c = 0$. First, we will look at $r = 0$, which reduces Equation 4.2 to the following:

$$(-1 + b)y = 0. \tag{4.12}$$

Looking at Equation 4.12 we see that the only two solutions are $b = 1$ and $y = 0$. When $b = 1$, Equations 4.3, 4.7, and 4.8 reduce to:

$$-ct = 0 \tag{4.13}$$

$$c(-p + z) = 0 \tag{4.14}$$

$$cy = 0. \tag{4.15}$$

Because $c = 0$ is a separate case, the only solutions to these three equations are $t = 0$, $p = z$ and $y = 0$. Therefore, a complete solution is $(a = 0, r = 0, b = 1, t = 0, p = z, y = 0)$.

Next, let $y = 0$ in Equation 4.12 instead of $b = 1$, then Equation 4.4 reduces to the following,

$$t - b^2 t = 0.$$

From here, the solutions are $b = 1$, $b = -1$ or $t = 0$. Because we have already have the solutions for when $(b = 0, y = 0)$, we will start with $b = -1$ which reduces equations 4.3 and 4.8 to the following,

$$-ct + 2x = 0$$
$$2n - cp + cz = 0.$$

When the above equations are equal to zero simultaneously, in addition to the previously set variables of $(a = 0, r = 0, y = 0, b = -1)$, then T is commutative.

Finally, let $t = 0$. This causes Equation 4.3 to reduce to

$$x - bx = 0.$$

This equation has the solutions $b = 1$ and $x = 0$. Because we have the solutions for when $(b = 1, t = 0)$ above, we will focus on $x = 0$. This causes equation 4.7 to reduce to

$$(-1 + b^2)m = 0. \tag{4.16}$$

There are 3 solutions to this equation, $b = 1$, $b = -1$, or $m = 0$. Once again, we already have the solutions for when $(b = 1, t = 0)$, thus we will start with $b = -1$. This gives us the following equation from Equation 4.8,

$$2n - cp + cz = 0.$$

Whenever this statement is true in addition to the previously determined variables of $(a = 0, r = 0, y = 0, t = 0, x = 0, b = -1)$, T is commutes with other matrices.

Next, when we return to the $m = 0$ case from Equation 4.16, we have the reduced form of Equation 4.8,

$$-bn + b^2n - cp + cz = 0.$$

Once again, whenever this statement is true along with the previously determined variables $(a = 0, r = 0, y = 0, t = 0, x = 0, m = 0)$, T commutes with other matrices.

Now, going back to Equations 4.10 and 4.11, we will set $c = 0$ instead of $r = 0$, which reduces Equation 4.7 to,

$$(-1 + b^2)m = 0.$$

The three solutions to equation above are $b = 1$, $b = -1$ or $m = 0$.

First, when we set $b = 1$ along with $a = 0$ and $c = 0$, then T simplifies to the identity matrix which commutes with anything. Therefore, $(a = 0, b = 1, c = 0)$ is a complete solution.

Next, let $b = -1$, which simplifies the matrix to only the Equations 4.2, 4.3, 4.6, and 4.8:

$$\begin{aligned} -2y &= 0 \\ 2x &= 0 \\ -2r &= 0 \\ 2n &= 0. \end{aligned}$$

Therefore, $(a = 0, b = -1, c = 0, x = 0, y = 0, r = 0, n = 0)$ is a complete solution.

Finally, when $m = 0$, the matrix reduces down to Equations 4.2, 4.3, 4.4, 4.6, and 4.8:

$$\begin{aligned} (-1 + b)y &= 0 \\ x - bx &= 0 \\ t - b^2t &= 0 \\ -(-1 + b)br &= 0 \\ (-1 + b)bn &= 0. \end{aligned}$$

Because we already did the $b = 1$ and $b = -1$ cases, we can ignore those solutions. Thus, a new solution is $(a = 0, c = 0, x = 0, t = 0, y = 0, r = 0, n = 0, m = 0)$. The last $a = 0$ case is when all of $a, b, c = 0$.

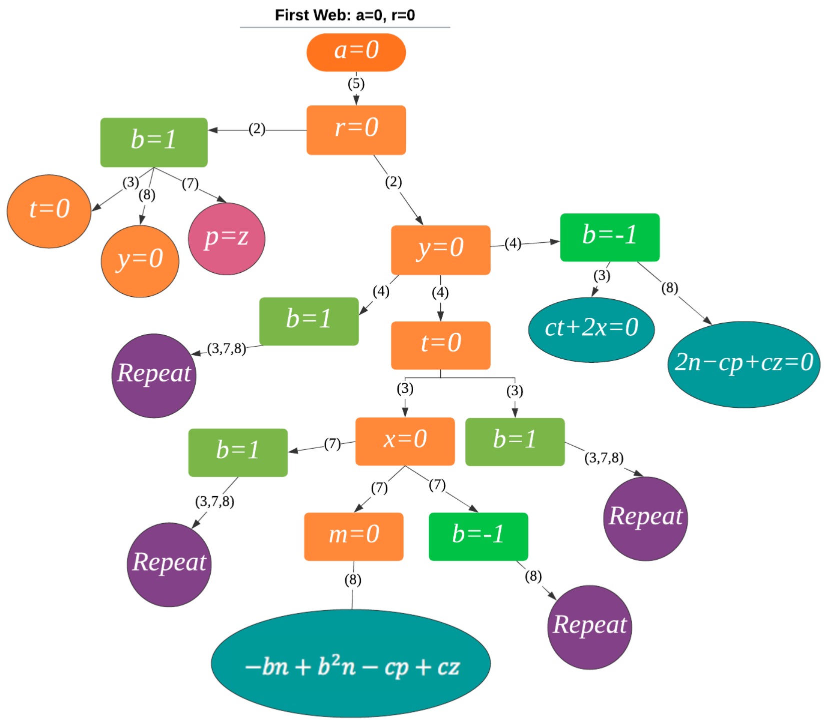

FIGURE 4.1. ***Web 1:*** *The $a = 0$, $r = 0$ Case*

The matrix reduces down to Equations 4.2, 4.3, 4.4, and 4.7:

$$
\begin{aligned}
-y &= 0 \\
x &= 0 \\
t &= 0 \\
-m &= 0.
\end{aligned}
$$

Therefore, a real solution is $(a = 0, b = 0, c = 0, x = 0, t = 0, y = 0, m = 0)$. ∎

4.2. $b = 0$ Case.

THEOREM 4.2. *If the matrix T commutes with other matrices when $b = 0$ then y and m from matrix D are zero.*

Proof. Let $b = 0$, which reduces Equation 4.2 to

$$-y = 0.$$

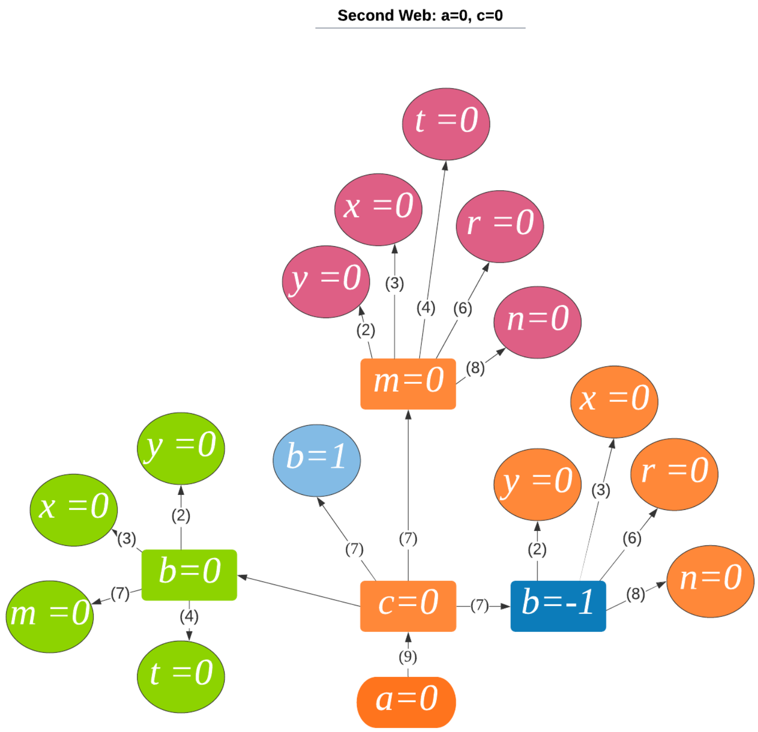

FIGURE 4.2. ***Web 2:*** *The* $a = 0$*,* $c = 0$ *Case*

Thus, we know that $y = 0$. When we evaluate y at 0, we get Equation 4.9 to reduce down to

$$a^2m = 0.$$

Because we did all the $a = 0$ cases, we know that $m = 0$ is the only new solution. When we evaluate $m = 0$, Equations 4.5 and 4.9 reduce to

$$cr = 0$$
$$-cr = 0.$$

From the equations above, either $r = 0$ or $c = 0$. First, let $r = 0$, which simplifies Equations 4.3, 4.4, and 4.8:

$$a^2n - ct + x + a(-w + z) = 0$$
$$t - 2act + a^2(p - w) = 0$$
$$c(2an - p + z) = 0.$$

Whenever the above equations are simultaneously equal to zero in addition to the previously determined variables of $(b = 0, y = 0, m = 0, r = 0)$, then the matrix T will commute.

Now, let $c = 0$, which allows us to determine that $(b = 0, y = 0, m = 0, c = 0)$. This gives us Equation

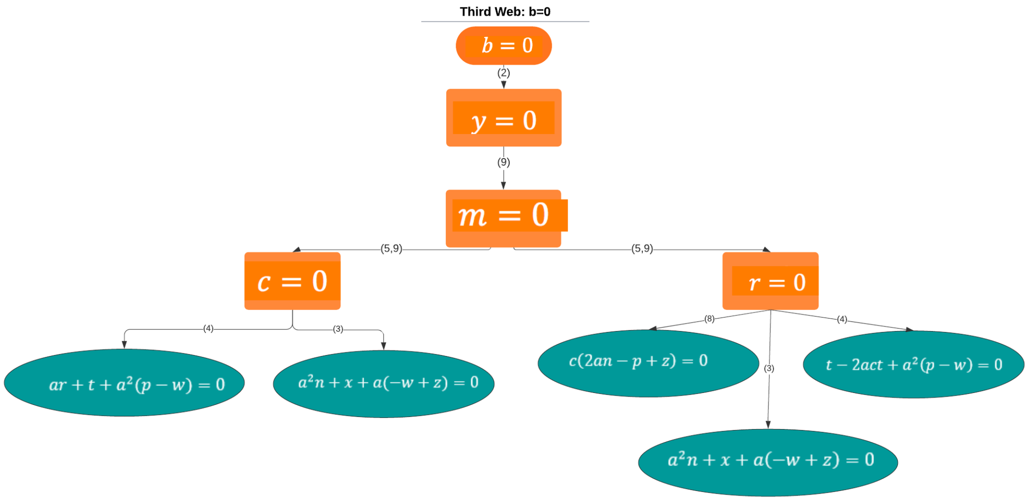

FIGURE 4.3. ***Web 3:*** *The* $b = 0$ *Case*

4.3 and 4.4 composed of the following terms:

$$a^2n + x + a(-w + z) = 0$$
$$ar + t + a^2(p - w) = 0.$$

When these two terms are simultaneously zero, then the matrix T commutes with other matrices. □

4.3. $c = 0$ **Case.**

THEOREM 4.3. *If the matrix* T *commutes with other matrices when* $c = 0$ *then* $m, y, n = 0$ *or* $b = -1$.

Proof. Let $c = 0$, which reduces entry 4.7 down to:

$$(-1 + b^2)m = 0. \tag{4.17}$$

From Equation 4.17, the only solutions are $m = 0$,$b = 1$, or $b = -1$. Beginning with $m = 0$, Equation 4.1 reduces down to:

$$ay = 0.$$

Because we already did all the $a = 0$ cases, we know that $y = 0$ is the only new solution. When we plug in

$y = 0$, Equation 4.5 reduces to:

$$2abn = 0.$$

Since we have already done both $a = 0$ and $b = 0$ cases, $n = 0$ is the only new solution. When we evaluate that, Equations 4.3, 4.4, and 4.6 reduce down to:

$$x - bx + a(-w + z) = 0 \tag{4.18}$$
$$t - b^2t + a^2(p - w) + a(r - 2bx) = 0 \tag{4.19}$$
$$b(r - br + 2a(p - z)) = 0. \tag{4.20}$$

When the above equations simultaneously equal zero, along with the previously determined variables ($c = 0, m = 0, y = 0, n = 0$), the matrix T commutes with other matrices.

Referring to the solutions from 4.17, we will now look at the $b = 1$ case, which gives us the following entries from Equations 4.2 and 4.8,

$$2am = 0$$
$$-am = 0.$$

The only new solution is $m = 0$, because we have already done $a = 0$ cases. This reduces the Equations 4.1 and 4.9,

$$ay = 0$$
$$-2an = 0.$$

These equations give the new solutions of $n = 0$ and $y = 0$. Thus, we can determine the variables ($c = 0, m = 0, n = 0, y = 0$) which give us Equations 4.18, 4.19, and 4.20, so there are no new solutions in this set, as any $b = 1$ cases are included in solutions already stated above.

Finally, referring to the solutions from equation 4.17, let $b = -1$:

$$-a(2n + y) = 0 \tag{4.21}$$
$$a(am + y) = 0 \tag{4.22}$$
$$-2(am + y) = 0 \tag{4.23}$$
$$-am + 2n = 0 \tag{4.24}$$
$$a(-am + 2n) = 0 \tag{4.25}$$
$$a^2n + 2x + a(-w + z) = 0 \tag{4.26}$$
$$-2ap - 2r - a^2y + 2az = 0 \tag{4.27}$$
$$a(r + a(p - w) + 2x) = 0. \tag{4.28}$$

Because we have already done the $a = 0$ case, we will solve each above equation for the variable a. When we solve the Equations 4.21 through 4.25 for a, we get

$$a = \frac{2n}{m}.$$

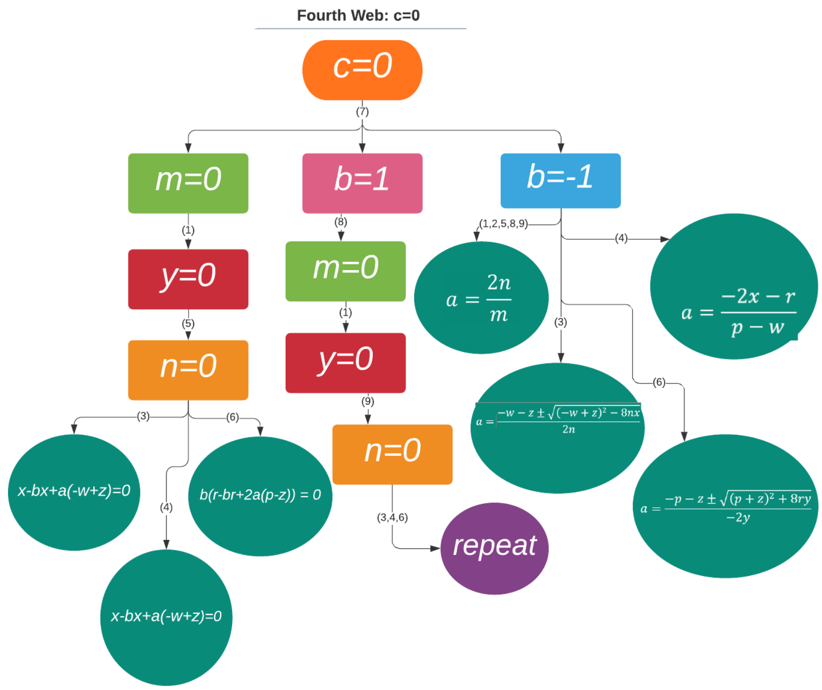

FIGURE 4.4. ***Web 4:*** *The* $c = 0$ *Case*

Next, when we put Equations 4.26 and 4.27 into the quadratic form,

$$na^2 + (-w + z)a + 2x = 0$$
$$-ya^2 - 2(p + z)a - 2r = 0.$$

We can use the quadratic formula to solve for a where

$$a = \frac{-w - z \pm \sqrt{(-w + z)^2 - 8nx}}{2n}$$

and

$$a = \frac{-p - z \pm \sqrt{(p + z)^2 + 8ry}}{-2y}.$$

Finally, when we solve Equation 4.28 for a,

$$a = \frac{-2x - r}{p - w}.$$

In order for the matrix T to commute, all of the above solutions for a would need to be equivalent simultaneously, in addition to the set variables $c = 0$ and $b = -1$. ☐

4.4. $a, b, c \neq 0$ Case.

THEOREM 4.4. *When a, b and $c \neq 0$, then $y = 0$ and $m = 0$.*

Proof. Let a, b, and c all be non-zero. Because $a \neq 0$, equation 4.1 has the solutions of $y = -am$ or $a = \frac{-y}{m}$.

First, we will prove by contradiction that

$$a = \frac{-y}{m} \tag{4.29}$$

is not a solution for the non-zero case. We will solve each of the 9 equations for a because T commutes with other matrices when each of the equations are simultaneously equal. Because we want $a \neq 0$, we know that $y \neq 0$ from equation 4.29. When we solve equation 4.2 for a,

$$a = \frac{y - by}{2bm}$$

and set this equation equal to equation 4.29,

$$\begin{aligned} -\frac{y}{m} &= \frac{y - by}{2bm} \\ -y &= \frac{y(1-b)}{2b} \\ -1 &= \frac{1-b}{2b} \\ -2b &= 1 - b \\ b &= -1. \end{aligned}$$

Thus, when we plug this solution in Equation 4.3 and solve for a,

$$a = \frac{y}{2m}.$$

When we set this equation equal to Equation 4.29,

$$\frac{y}{2m} = -\frac{y}{m} \tag{4.30}$$

we see that the only solution would be $y = 0$, which would make $a = 0$, thus $a = \frac{-y}{m}$ is not a solution for when a, b and c are not zero.

Next, we will look at

$$y = -am \tag{4.31}$$

from Equation 4.1. Because there are other equations with the variable y, we must solve each equation for y and find when they are simultaneously equal. When we solve Equation 4.2 for y,

$$y = \frac{-2amb}{-1+b}.$$

When we set this equation equal to Equation 4.31,

$$\begin{aligned}-am &= \frac{-2amb}{-1+b} \\ (-1+b)(-am) &= -2amb \\ -1+b &= 2b \\ -1 &= b.\end{aligned}$$

Because $a \neq 0$, our two solutions are $b = -1$ or $m = 0$. Next, when we solve Equation 4.3 for y,

$$y = \frac{-(-1+b^2-2ac)m}{c}.$$

When we plug in $b = -1$ and set this equal with Equation 4.31,

$$\begin{aligned}-am &= \frac{-(-1+1-2ac)m}{c} \\ -am &= 2am.\end{aligned}$$

Because $a \neq 0$, it must be true that $m = 0$. Therefore, $m = 0$ is the only solution that allows all three equations to be equal. This proves that both y and m equal zero, as $y = 0$ when $m = 0$.

Now, let $y = 0$ and $m = 0$

$$2abn - cr = 0 \tag{4.32}$$

$$-2abn + cr = 0 \tag{4.33}$$

$$\begin{aligned}a^2n - ct + x - bx + a(-w+z) &= 0 \\ t - b^2t + a^2(p-w) + a(r - 2(ct+bx)) &= 0 \\ -(-1+b)br - 2a(cr + b(-p+z)) &= 0 \\ -bn + b^2n + c(2an - p + z) &= 0.\end{aligned}$$

Because Equations 4.32 and 4.33 are equivalent, there are only five unique equations that must all equal zero in order for the matrix T to commute when $y = 0$ and $m = 0$. This completes the proof. ∎

5. Further Work.

1. We were unable to find final solutions in the unitary case. We would need to find A so that we are able to solve for a, b and c. This would confirm all solutions for the 3×3 binormal composition matrix.
2. We started work with the 4×4 binormal solutions, but could not confirm we had all answers. Are there any non-zero binormal solutions?
3. Similarly, can we prove that all binormal matrices are also n-normal?
4. Are there any general commutative cases that will work for any $n \times n$ matrix?

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