INTUITIONISM

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Derives philosophically from Kant's Conceptualism -- the objects of mathematical knowledge only have reality within the mind, they do not have any reality apart from our thinking.

Main Protagonists: Kronecker, Poincaré, Borel,
Brouwer, Weyl, Heyting

Not every imagined object exists in reality. Therefore, the creative powers of the mind must be bound by strict limitations:

[1] The notions of finite set and of real numbers should not be based upon the idea of an already existing totality
[2] Definitions and proofs should be constructive
[3] Anything that is admitted to existence must be constructable in finitely many steps
[4] Formal logical consistency is required
[5] Impredicative definitions are rejected since they presuppose the existence of that which is supposedly being generated
[6] The Law of Excluded Middle (i.e., a statement is either true or false) is limited except in obvious cases

Numbered sections 1 - 6 which follow expand upon the above six restrictions.

1] The notions of finite set and of real numbers should not be based upon the idea of an already existing totality.

A. An object exists only by being the result of a construction.

Define the naturals as the constructable collection consisting of
i) some beginning set {1,2,3,4}, and
ii) a rule for deriving new elements 4+1=5, 5+1=6, 6+1=7, ...

then 21, 376, 498, 320, 675, 542 exists not because of some assumed existence in a predefined totality but because it can be generalized by the above construction.

B. Cantor's theory of transfinite numbers is invalid.
It begins with the assumption that infinite sets exist.

2] Definitions and proofs should be constructive

A. Reductio ad absurdum existence proofs are unacceptable

i) <u>Fundamental Theorem of Algebra</u>--within the complexes are all roots of polynomials over the complexes
Typical proof: Assume existence of an algebraic equation with no roots. Obtain constradiction.

Such proofs give no rule for finding a root.
∴ logical contradiction does not imply existence

ii) a. Cantor's proof that transcendental numbers exist is invalid.

Suppose there are no transcentals. Since the reals (and therefore the complexes) are nondenumerable, while the algebraic numbers are denumerable, there must exist non-algebraic complexes.

b. Louiville in 1851 gave a constructive proof

c. Lindemann's proof of 1882 that π is transcendental elicited from Kronecker the response, "That's interesting, excepting for the fact that π doesn't exist."

B. Existence proofs which rely on the axiom of choice are rejected.

Well-ordering Theorem--every set may be well-ordered
Proof assumes that by the choice axiom for each $S' \subset S$ there is a representative element $x(S')$.

No rule for specifying $x(S')$ is given.

3] Anything that is admitted to existence must be constructable in finitely many steps.

A. Finite induction is permitted

What does it mean that property p holds for all naturals?
Given any natural n, p holds for each element in the set $\{1,2,3,\ldots, n\}$

B. Cantor's argument that there exist more reals than naturals is unacceptable.

Suppose $\exists$ 1-1 correspondence between the naturals and (0,1). Write each real r_1, r_2 $r_3\ldots$ in decimal form:

$$r_1 = .a_{11}a_{12}a_{13}a_{14}\ldots$$
$$r_2 = .a_{21}a_{22}a_{23}a_{24}\ldots$$
$$r_3 = .a_{31}a_{32}a_{33}a_{34}\ldots$$
$$\vdots$$

Form $r_0 = 0.a_{01}a_{02}a_{03}\ldots$

Rule $a_{on} = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 0 & \text{otherwise} \end{cases}$

Then $r_o \neq r_i \ \forall i \in N$

But such a construction of r_o requires infinitely many steps.

4] Formal logical consistency is required

Law of Non-contradiction accepted
i.e., Propositions p and $\sim p$ are not both true

Self-contradictions may not be brought into existence
e.g., consider fraction $\frac{p}{q}$ with nonrepeating decimal representation.

5] Impredicative definitions are rejected since they presuppose the existence of that which is supposedly being generated.

A. Let A be the set of all sets

In order for A to be well defined, each of its elements must be known.

But $A \in A$, making the definition circular.

B. The Axiom of Completeness

For every bounded set S of real numbers, $\exists$ set U of upper bounds of S in which there exists $U_o \ni U_o \leq u \ \forall u \in U$

U_o must exist as an $U.B.$ before U_o can exist as $L.U.B.$

6] The Law of Excluded Middle (i.e., a statement is either true or false) is limited except in obvious cases.

A. The statement

"$\{x \mid x \in N \wedge a < x < b$ for fixed $a, b \in N\}$ contains a nonempty set of twin primes"

is true or false (with no other possibility) since only finitely many pairs would have to be checked

B. The Twin Prime Conjecture that there exist infinitely many twin primes may be neither true nor false.

C. The Consequence of Excluded Middle that $\sim(\sim p) \rightarrow p$ is rejected

e.g., Let p be the proposition that $\sqrt{2}$ is irrational assuming the negation of p would assent

$$\exists\, m,n \ni \frac{m}{n} = \sqrt{2} \text{ with } (m,n) = 1$$

but then $m^2 = 2n^2$ which implies $2/m$.

Let $m = 2r$

then $4r^2 = 2n^2$ which implies $2/n$

thus $(m,n) = 2$, a contradiction.

Conclusion: $\sim p$ is false, and $\sim(\sim p)$ is true.

The intuitionist would not say $\sim(\sim p)$ implies p
$\sim(\sim p)$ might be some third possibility.

SUMMARY
STRENGTHS OF INTUITIONISM

1] Many paradoxes are easily discharged

A. The Russell Paradox built on the set of all sets not containing themselves as members

S does not exist. Its definition is non-constructive and the source of contradiction (see limitations 2,4)

B. Epimenides, the Cretan, stated that all Cretans always lie. Was his statement true or false?

If true, then Epimenides lied, and his statement was false
If false, then Epimenides lied, and in so doing he spoke truly

Since both assumptions lead to contradictions, the statement must be neither true nor false. (see limitations 4,6)

2] The restrictions and constructive methods of intuitionism do not lead to contradictions.

3] Decidability problems seem to support limiting the use of Excluded Middle.

If Excluded Middle is universally valid, then every problem is solvable. But if not universally valid, there may be some that are not solvable.

Goldbach's Conjecture - Every even number greater than 2 is the sum of two primes.

The Twin Prime Conjecture - $\exists$ infinitely many twin primes