Using *Mathematica* to Teach Calculus

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1. INTRODUCTION:

For the past two years Westmont College has been one of the beta test sites for the calculus reform experiment being conducted at the University of Illinois under the direction of Jerry Uhl. Brown, Porta, and Uhl have created text which is integrated with *Mathematica*, a very powerful symbol manipulation, graphics, and number crunching software package produced by Wolfram Research, Inc. A preliminary version of this text has just been released [2]. We have used the Illinois materials for an honors course of incoming Freshmen with prior calculus experience. The purpose of this paper is to evaluate the curriculum and illustrate how we have used it. Hopefully this will be helpful to those who are considering using *Mathematica* in their instruction.

2. A BRIEF LOOK AT *MATHEMATICA*:

2.1: *Number Crunching*:

The numerical precision of *Mathematica* is limited only be the amount of memory one has available. The following examples serve to illustrate this: The command

`N[2^(1/5),10]` produces a ten digit representation of the fifth root of two:

```
1.148698355
```

`N[Pi,10]` yields

```
3.141592654
```

Pi fetishes may want to try

`N[Pi,1000]` to get

```
3.1415926535897932384626433832795028841971693993751058209749445923078164062862\
  08998628034825342117067982148086513282306647093844609550582231725359408128\
  48111745028410270193852110555964462294895493038196442881097566593344612847\
  56482337867831652712019091456485669234603486104543266482133936072602491412\
  73724587006606315588174881520920962829254091715364367892590360011330530548\
  82046652138414695194151160943305727036575959195309218611738193261179310511\
  85480744623799627495673518857527248912279381830119491298336733624406566430\
  86021394946395224737190702179860943702770539217176293176752384674818467669\
  40513200056812714526356082778577134275778960917363717872146844090122495343\
  01465495853710507922796892589235420199561121290219608640344181598136297747\
  71309960518707211349999998372978049951059731732816096318595024459455346908\
  30264252230825334468503526193118817101000313783875288658753320838142061717\
  76691473035982534904287554687311595628638823537875937519577818577805321712\
  2680661300192787661119590921642.
```

2.2. *Symbol Manipulation*:

`Expand[(x+2y)^13]` produces

$$x^{13} + 26\,x^{12}\,y + 312\,x^{11}\,y^{2} + 2288\,x^{10}\,y^{3} + 11440\,x^{9}\,y^{4} +$$
$$41184\,x^{8}\,y^{5} + 109824\,x^{7}\,y^{6} + 219648\,x^{6}\,y^{7} + 329472\,x^{5}\,y^{8} +$$
$$366080\,x^{4}\,y^{9} + 292864\,x^{3}\,y^{10} + 159744\,x^{2}\,y^{11} + 53248\,x\,y^{12} + 8192\,y^{13}$$

One also has the capability to define functions. (A space between tokens indicates multiplication):

`f[x_] := 5x Sin[x^2] Log[x^2-7]` and then differentiate:

`f'[x]`

```
                                      2       2
      2        2                2    10 x  Sin[x ]              2         2
  10 x  Cos[x ] Log[-7 + x ] + ------------- + 5 Log[-7 + x ] Sin[x ]
                                          2
                                    -7 + x
```

`f''[x]`

```
      3        2                                      3        2
   40 x  Cos[x ]                 2              2     20 x  Sin[x ]
 ((------------- + 30 x Cos[x ] Log[-7 + x ]) - ------------- +
           2                                            2 2
     -7 + x                                       (-7 + x )

            2
    30 x Sin[x ]        3            2        2
    ------------) - 20 x  Log[-7 + x ] Sin[x ]
           2
      -7 + x
```

The differential operator, **D**, can be used to compute derivatives also:

`D[5x^3 - 4b x^2 + 13c x + 7, x]`

```
                       2
  13 c - 8 b x + 15 x
```

The integration routines are powerful. Consider the following problem which appeared in the April 1988 issue of Mathematics Magazine: *Express* $\int \sqrt{\tan x}\,dx$ *in closed form.* Although the editors probably would not accept this, one could try submitting:

`Integrate[Sqrt[Tan[x]],x]`

```
                    -Sqrt[2] + 2 Sqrt[Tan[x]]
  Sqrt[2] ArcTan[-------------------------]
                             Sqrt[2]
  ----------------------------------------- +
                         2

                    Sqrt[2] + 2 Sqrt[Tan[x]]
    Sqrt[2] ArcTan[------------------------]
                             Sqrt[2]
    ---------------------------------------- +
                         2

    Sqrt[2] Log[1 - Sqrt[2] Sqrt[Tan[x]] + Tan[x]]
    ---------------------------------------------- -
                          4

    Sqrt[2] Log[1 + Sqrt[2] Sqrt[Tan[x]] + Tan[x]]
    ----------------------------------------------
                          4
```

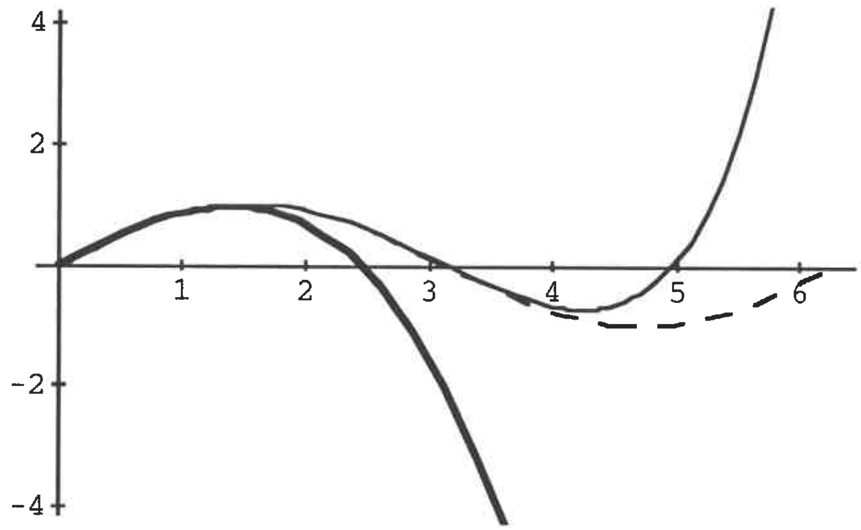
2.3. *Graphics*:

2.3.1. 2 *Dimensions*:

The command

```
Plot[{Sin[x],
      x - x^3/3!,
      x - x^3/3! + x^5/5! - x^7/7! + x^9/9!},
      {x,0,2Pi},
      PlotStyle->{Dashing[{0.03,0.03}],
                  Thickness[0.005],
                  Thickness[0.001]}]
```

produces a plot of the sine function (dashed), and the Taylor polynomials of degree two (thick) and five (thin) in the following figure:



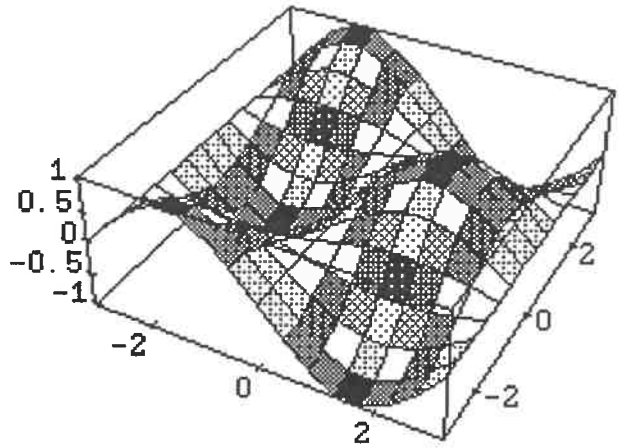
The capability exists to expand or shrink the graphics display, as well as to obtain (x,y) coordinate positions which change as the student moves the mouse within the picture. It is not hard to imagine how useful something like this can be in the teaching of Taylor series.

2.3.2. 3 *Dimensions*:

The command

```
Do[Plot3D[Sin[x+n]  Cos[y+n],
        {x,  -Pi,  Pi},  {y,  -Pi,  Pi}],
        {n,  0,  Pi-Pi/16,  Pi/16}]
```

generates a series of 16 plots of sin(x+n)cos(y+n), where x and y both range between $-\pi$ and π, and n ranges from zero to $\frac{15\pi}{16}$ in increments of $\frac{\pi}{16}$. The following page gives the plot obtained when n = 0:



By grouping the sixteen plots together, a movie-like animation can be made illustrating how the graph changes in a dynamic fashion.

3. AN OVERVIEW OF THE BROWN, PORTA, UHL TEXT:

3.1. *Format*:
The text itself comes with 6 disks in a 3-ring binder, along with a paperback printout of the bulk of the contents of the disks. Included in this printout are a section of literacy sheets, designed to test student knowledge away from the computer. A typical chapter is set up as follows:

The Race Track Principle

Guide
Basics
Tutorial
Give it a try

Literacy Sheet (Hardcopy only)

The *Guide* section gives an overview of the chapter, while the *Basics* section maps out the general principles upon which the chapter is based. Applications of these principles are found in the *Tutorial* section, and *Give it a try* contains problems.

Each of the italicized sections above can be expanded by clicking the mouse at an appropriate place. The selected section will then expand, as if one were turning to a specific part of a book. For example, clicking on the ***Basics*** section gives the following:

B.1) ***The Race Track Principle.***
B.2) ***Logarithmic and Exponential Growth.***
B.3) ***The Mean Value Theorem.***

Of course, each of the above sections can be further expanded. Since, a full printout of the sections comes with the text, students do not need access to a computer if they wish only to read the material.

3.2. *General Approach*:
The authors list four explicit goals in their preface, stating that they wish to promote:

1. Active learning through experimentation.
2. A conceptual emphasis on the process of problem solving by using the calculational power of *Mathematica* so that, "...students can work with real data."
3. Geometric insight, gained by having students view plots of difference quotients, watching trapezoidal approximations converge to the value of an integral as the mesh approaches zero, etc.
4. Mathematical maturity (perhaps better stated as mathematical intuition), developed by students seeing Taylor polynomials closing in on the function they are approximating, and by returning again and again to the Fundamental Theorem which, "...is restored to a central role assigned to it by Isaac Newton as the calculational base for measurements made by integration."

The experimental goal mentioned above tends to promote a *loose* approach in the derivation of many traditional ideas. For example, the authors' proof that the derivative of $f(x) = \sin x$ is $\cos x$ consists in plotting difference quotients $\left(f(x+h) - f(x)\right)/h$ for smaller and smaller values of h over a range of x values. After comparing those graphs with the cosine function, they state, "The evidence tells us that the limiting case of the average growth rates...as h closes in on 0 is $f'[x] = \cos[x]$."

The authors are *careful*, however, in the application of fundamental ideas. The notion that the integral of a derivative is the original function is repeatedly emphasized, and the typical method of using mere symbol pushing (without understanding) to solve separable differential equations is carefully avoided. One advantage of having something like *Mathematica* available is that the capability exists to experiment with mathematical modeling early. Students may know they want to integrate something (e.g. $\cos^2 x$) with no idea at a particular stage of the course how to do so. With the aid of *Mathematica*, the idea of the Fundamental Theorem can be reinforced without worrying about things such as half-angle formulas getting in the way.

Fundamental to much of the perspective in the Brown, Porta, Uhl text is the notion of the derivative as a rate of change. This comes in quite handy in many applied contexts. There is some, albeit weak, treatment of the derivative as the slope of that tangent line. Instructors wishing to use this text might consider expanding the treatment that is given to that point.

3.3. *Topics Covered*:
Brown, Porta, and Uhl have created a bold "lean and lively" calculus curriculum. For better or worse, many of the topics traditionally covered in a calculus course are missing. The notions of continuity, concavity, and what the graph of a differentiable function looks like are never explicitly mentioned. The traditional sections on integrating powers of sine and cosine, tangent and secant, are ignored. The quotient rule is not mentioned. Instead, the chain rule and logarithms are employed to compute most derivatives, unless *Mathematica* is called on for this purpose.

The power of the computer is used to crank out specific instances of recursive reduction formulas which the students derive ahead of time. They enjoy working out the formulas (the important part) and watching the computer generate specific solutions (the tedious part). For example, the formula

$$\int \cos^n(x)dx = \tfrac{1}{n}\cos^{n-1}(x)\sin(x) + \tfrac{n-1}{n}\int \cos^{n-2}(x)dx$$

can easily be modeled by *Mathematica* as follows:

```
i[0] = x
   x

i[1] = Sin[x]
   Sin[x]

i[n_] := 1/n Cos[x]^(n-1) Sin[x] + (n-1)/n i[n-2]
```

Once the recursive function has been defined, it is a trivial matter to evaluate integrals involving powers of the cosine function:

```
i[2]
   x   Cos[x] Sin[x]
   - + -------------
   2         2

i[5]
                                            2
                            2 Sin[x]   Cos[x]  Sin[x]
           4            4 (-------- + --------------)
   Cos[x]  Sin[x]             3              3
   -------------- + ----------------------------
         5                       5


Simplify[%]          (% refers to the last output.)
                     2            4
   (8 + 4 Cos[x]  + 3 Cos[x] ) Sin[x]
   ----------------------------------
                  15
```

The syntax for *Mathematica* can get in the way at times. Note that function evaluation is handled via *square brackets*. When a function is defined, its arguments must be indicated by following them with an underscore. All built in functions supplied by *Mathematica* begin with a capital letter. Note also the difference between the = and the := symbols. The former forces function evaluation immediately, while the latter evaluates the function only when it is called with specific arguments. Thus, not to define **i[n_]** above with the := symbol would send the machine into an infinite recursive loop.

Many ideas not heretofore practicable to be covered are treated in the Brown, Porta, Uhl text. Students will explore Pade's approximation techniques, Fourier series, least square fits, and interpolating polynomials. They learn that the derivative of ln(x) is 1/x very early, and also study complex numbers, partial derivatives, and multiple integrals. Bill

Davis at The Ohio State University is currently developing a third-semester calculus text using *Mathematica.*

4. EVALUATION:

4.1. *Use of Mathematica:*

The symbol manipulation capability is used frequently, but perhaps overly so. For example, students see the machine simplify the difference quotient for $f(x)=x^2$ rather than derive this on their own. The intersection of the curves y = x with $y=x^2$ is also obtained by machine. Other non-necessary uses of symbol manipulation abound.

The graphics capability is used extensively and well. It is hard to imagine a student getting through this type of a course without a good feel for how graphs of different functions behave. By contrast, the animation capability is not exploited to its full extent. Arguments against using animation focus on the amount of memory required. However, if one's attention is restricted to two dimensional graphs, most animations will fit nicely in the five megabytes of memory recommended for using the software in the first place. Instructors wishing to include more animations may do so, and in fact may change the text in any way they wish, provided they so note the changes they have made at the appropriate place in the text.

4.2. *Exercise Sets*:

The exercises are practical and imaginative. Since students can use the machine to "do the nasty work," many real-life examples are explored. The exercises encourage experimentation, but it is difficult to to tie them into specific sections that have been studied in a given chapter. In addition, there is a notable lack of theoretical questions which instructors might want to assign to challenge the better students. The headings of the exercises (from the "Give it a try" section) given at the end of one of the differential equations chapters are given below, with one of the cells (G.4) expanded:

G.1) Separating and integrating.
G.2) More population predictions.
G.3) A modification of the predator-prey model.
G.4) Spread of infection model: The logistic equation.
G.4.a)

```
   Suppose we have a closed population of P equally suspectible infection-free
individuals and  that one additional infected individual is introduced . Let y[t]
be the number of infected individuals at time t and let x[t] be the others.
   We have
          x[t] + y[t] = P+1.
   Why is it reasonable to assume that the rate y'[t] at which the infected
population is proportional to both y[t] and to x[t]?
   Why does this assumption result in the relationship
          y'[t]  = k y[t](P + 1 - y[t])?
```

G.4.b)

```
   Separate the variables and integrate under the condition y[0] = 1 .
```

G.4.c)

```
   Find the value of k under the assumption that P =1000 and at time t = 5, we
know that 200 individuals are infected.
   Then plot y[t].
   How large is y[t] when the infection is spreading most rapidly?
```

The following discussion is adapted from an article, "Modeling the AIDS epidemic" by Allyn Jackson in the October, 1989 issue of the Notices of the American Mathematical Society, pp.981-983.

From Jackson's article:

"A team of mathematicians at Los Alamos National Laboratory has been using sophisticated mathematical techniques to formulate a set of models of the AIDS epidemic. . .One of the most surprising aspects of the AIDS epidemic is that, unlike most epidemics, it does not exhibit (the logistic pattern discussed above). Rather the best fit to the data turns out to be a cubic polynomial. The Los Alamos team started out with the logistic equation(the same equation you studied above)... The team built their model by successively modifying this equation to accommodate the various parameters...The Los Alamos group is involved in a project to develop user-friendly software that will allow public health workers to use the model to better understand the future of the epidemic. For this project, researchers at the University of Illinois are working on the software, the Census Bureau is providing data and the Air Force Academy will be testing the package..."

G.5) Chemical reaction model.
G.6) Combat model: Conventional warfare.
G.7) Combat model: Colonial vs. guerilla warfare.
G.8) Diving and calculus.
G.9) Pressure altimeters.
G.10) Forensic medicine.

(See [2, "Give it a try," pp. 169-179]).

4.3. *Writing Style*:
The text is clearly written, and as can be observed by reading over the section above, contains the normal amount of typos one might expect to find in a preliminary edition. The authors write in a koine style. In one sense this is appealing as the book does not appear to be stuffy. On the other hand, there are many times when the language is inappropriate. Instructors adapting this text may wish to edit those parts.

4.4. *Required Resources*:
The Brown, Porta, Uhl text is currently only available for Macintosh computers. A machine with at least a 68030 processor (e.g. an old Macintosh SE30, or a Macintosh IIcx), a hard drive, and 5 MB of memory is recommended. Color monitors are not necessary, but enhance the visual display enough to make their purchase worth it. To run this course successfully, one machine for every two students is recommended. The cost of the *Mathematica* software is expensive, listing at $795. However, grant programs exist, both from outside sources and from Wolfram Research, Inc. The software is available for students (*not* the enhanced version, thus it runs more slowly) at a price under $200.

5. USE AT WESTMONT COLLEGE:

5.1. *General Setup and Student Reaction*:
At Westmont College, we use the course for Freshmen with prior calculus experience. The group enrolled last Fall came to the college with a high school g.p.a. of 3.77, and average SAT scores of 600 (verbal) and 680 (mathematical). We take two weeks of the course to review first semester calculus in conjunction with learning the *Mathematica* and Macintosh systems. The remainder of the course covers second semester calculus material, with some curricular alterations from the standard topics.

Students uniformly responded favorably to the course. Initially, there was some frustration in getting comfortable with the computer system. A student comment illustrates this: "I really struggled with this class the first few weeks because it was hard to adjust to the different classroom style. It was hard for me to understand what role the computer was supposed to play. Was it supposed to teach me or just be a resource for help? Or as a calculator? Once I realized to listen to the lectures and just apply that to the problems, using *Mathematica* as a resource for examples and calculations, I did much better."

Other student comments ran along the following lines:

"I felt *Mathematica* was very helpful in understanding the concepts, especially the ones which included graphing--of which there were many."

"The computers were great, when the printer worked. I liked *Mathematica* a lot. It helped us go further into the material."

"I really like the *Mathematica* program. I definitely feel that we were able to do more advanced, real-life problems, like least squares fit, log-log, and other things like this."

Since the group which used the program consisted of above-average students, we have no data which would help assess how successful this curriculum might be for typical beginning calculus students.

5.2. *Pedagogical Changes. A Different Approach to Teaching Taylor Series*:
One advantage to having computerized texts is that instructors using them have the capability to modify those parts they wish to change. It is this writer's opinion that the standard approaches to Taylor series lack motivation, and from an intuitive point of view leave most students "scratching their heads." In a forthcoming article, [1] Howell and Mathews show how a computer algebra system can assist in a different approach to the teaching of Taylor series. A brief sketch of their ideas is given below:

It is well-known that the tangent line $p_1(x) = f(x_0) + f'(x_0)(x - x_0)$ is the limiting case as h approaches zero of the secant line drawn between the points $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$. The slope of this line is, of course, the coefficient of the linear term in the Taylor expansion of f(x). It is not as well-known that this phenomenon occurs for higher degree polynomial approximations as well. What would happen, for example, if we took a "secant parabola" passing through the *three* points $(x_0, f(x_0))$, $(x_0 + h, f(x_0 + h))$, and $(x_0 + 2h, f(x_0 + 2h))$ and let h approach zero? It is pleasing to find out that the limiting "tangent parabola" has the standard Taylor coefficients in its equation: $p_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$. Using *Mathematica*, some of the tedious algebra involved in showing this can be derived quite easily, and the animation capability can be used to show what happens to various graphs as h closes in on zero. A brief summary of how this can be done follows:

We start with the standard formula for Newton's polynomial of degree 2:

```
EQ = y == a0 + a1(x-x0) + a2(x-x0)(x-x0-h)
y == a0 + a1 (x - x0) + a2 (x - x0) (-h + x - x0)
```

Then we Substitute $(x_0, f(x_0))$, $(x_0+h, f(x_0+h))$, and $(x_0+2h, f(x_0+2h))$ into the equation to get a lower-triangular system:

E1 = EQ/.{x->x0,y->f[x0]};
E2 = EQ/.{x->x0+h,y->f[x0+h]};
E3 = EQ/.{x->x0+2h,y->f[x0+2h]};
TableForm[{E1,E2,E3}]

```
f[x0] == a0

f[h + x0] == a0 + a1 h

                                     2
f[2 h + x0] == a0 + 2 a1 h + 2 a2 h
```

Mathematica can easily solve this system for the coefficients:

Solutionset = Solve[{E1,E2,E3}, {a0,a1,a2}];
Solutionset = First[MapAll[Together, Solutionset]]

```
                          -f[x0] + f[h + x0]
{a0 -> f[x0], a1 -> ------------------,
                                  h

           f[x0] - 2 f[h + x0] + f[2 h + x0]
  a2 -> ---------------------------------}
                                  2
                               2 h
```

The "secant polynomial" is given by substituting these coefficients into the Newton Polynomial **EQ**.

p2[x_] = EQ[[2]]/.Solutionset (The [[2]] refers to the part of **EQ** after the == in its definition above.)

```
           (x - x0) (-f[x0] + f[h + x0])
f[x0] + ----------------------------- +
                         h

  (x - x0) (-h + x - x0) (f[x0] - 2 f[h + x0] + f[2 h + x0])
  ----------------------------------------------------------
                                   2
                                2 h
```

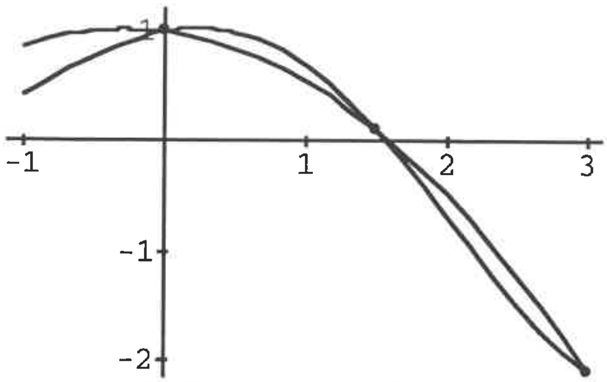
Mathematica can compute the symbolic limit of the above polynomial. (In version 2.0, the *analytic* option must be set to *true* to insure the limit is defined.)

Limit[P2[x], h->0]

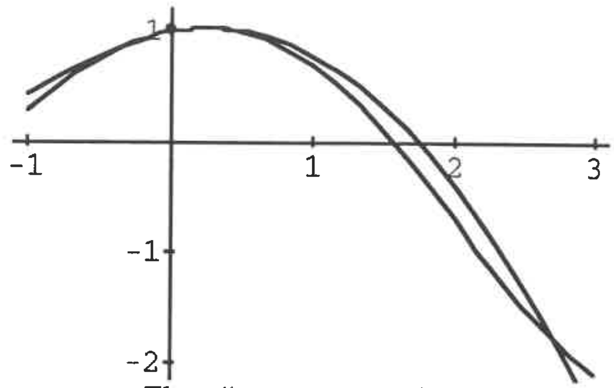
```
                                           2
                                  (x - x0)   f''[x0]
f[x0] + (x - x0) f'[x0] + -----------------
                                      2
```

EUREKA!

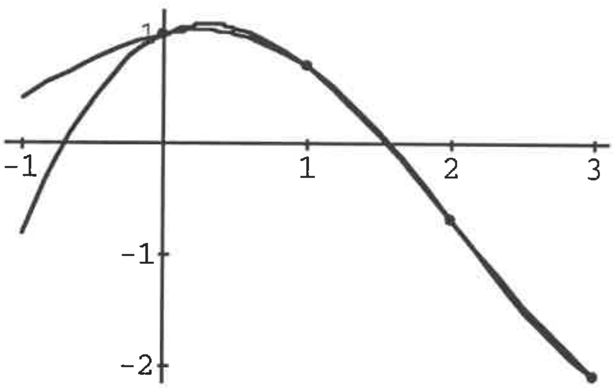
The figures below give the first and last frames of an animation showing "secant" and corresponding "tangent" polynomials of degree 2 and 3 for $f(x) = e^{\frac{x}{4}} \cos x$ with $x_0 = 0$.



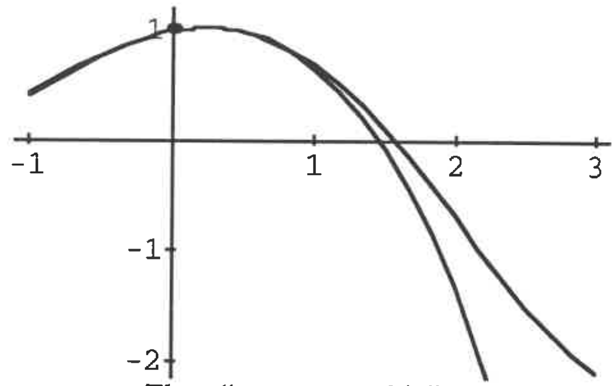
A "secant parabola"



The "tangent parabola"



A "secant cubic"



The "tangent cubic"

Once students understand the case for polynomials of degree 2 and 3, generalizing to polynomials of higher degrees should be an easy matter to negotiate.

6. CONCLUDING REMARKS:

For several years we have experimented with different curricula for students having had calculus in high school. Teaching a theoretical "honors" class was not satisfactory, neither was the placement of such students in a sophomore level course. The approach we have tried using *Mathematica*, however, seems to be working out quite well. It is new enough so that the students do not think they have seen the material before (as was the case with our honors course), and flexible enough to present ideas at various levels. Becoming proficient at using the computer is not a trivial matter, however, and it is not clear how easily the average calculus student would adjust to such a system.

REFERENCES:

[1] Howell, R.W., and Mathews, J., Investigation of Tangent Polynomials with a CAS, *The AMATYC Review*, (forthcoming).

[2] Brown, D., Porta, H., and Uhl, J., *Calculus and Mathematica (Part I, Preliminary Edition)*, Addison-Wesley, 1991.