Tracking the Trochoid on Safari

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This paper is really two papers in one. One of them is a discussion of the trochoid family of curves, presenting them in a novel context which should prove of use to those who regularly teach calculus classes or who simply like lovely curves. The other is a discussion of life in East Africa with special emphasis on mathematics education at the university level, a topic which may be of general interest to everyone at the conference. The only real connection between these two topics is that my work on trochoids as presented here was mostly done in Africa. Analogies between the two abound however. For example, during a three day safari in the great grasslands of the Serengeti Plain, our guide never ceased to point out animals whose presence would otherwise have remained "unspotted". The yellow, browns and greens of the grasses well camouflage the yellows and browns of, say, the cheetah. The guide knew where to look; the very arrangement of the herds of zebra and wildebeest oftentimes are clues to where a predator lurks. In somewhat the same manner, mathematical objects lie hidden; and various indicators mark their presence. So this paper is akin to a little safari on which we hope to spot the trochoid hidden in an envelope of sorts. But first a short interlude on Africa.

Life at an African University

On two occasions I had the honor of serving as a Fulbright professor in Africa: at the University of Botswana in Gaborone, Botswana, during the 1990-91 academic year and at the University of Dar es Salaam in Tanzania during the 1997-98 academic year. Let's focus on Tanzania. Figure 1 is a crude map of the country which doubled as a Christmas greeting during the year of our adventure.

The little house on the map respresents our home in a bustling seaport of five million people. Offshore is the island of Zanzibar, once a major clearinghouse of slavery in past centuries and still the place from which a significant portion of the world's spices is grown. The skull marks Olduvai Gorge, an excavation site on early man. The mountains are the double peaks of Kilimanjaro, tallest in Africa. Between the Mountain and Lake Victoria, the headwaters of the Nile, is the Serengeti, home to great herds of grazers ever roving, seeking fresh grass. The flamingo straddles the small, genocidal-torn countries of Burundi and Ruwanda, the Tanzanian borders of which yet flare with militia activity. Where the flamingo steps is the place where Stanley met the famed missionary-explorer, using the words, "Dr. Livingstone, I presume." The elephant stomps in the Selous, the largest game reserve in the world. In the four wheel drive vehicle is my family, bumping over potholes and narrow tracks, exploring this land of 120 different tribes totalling 26 million people in a territory half again the size of Texas.

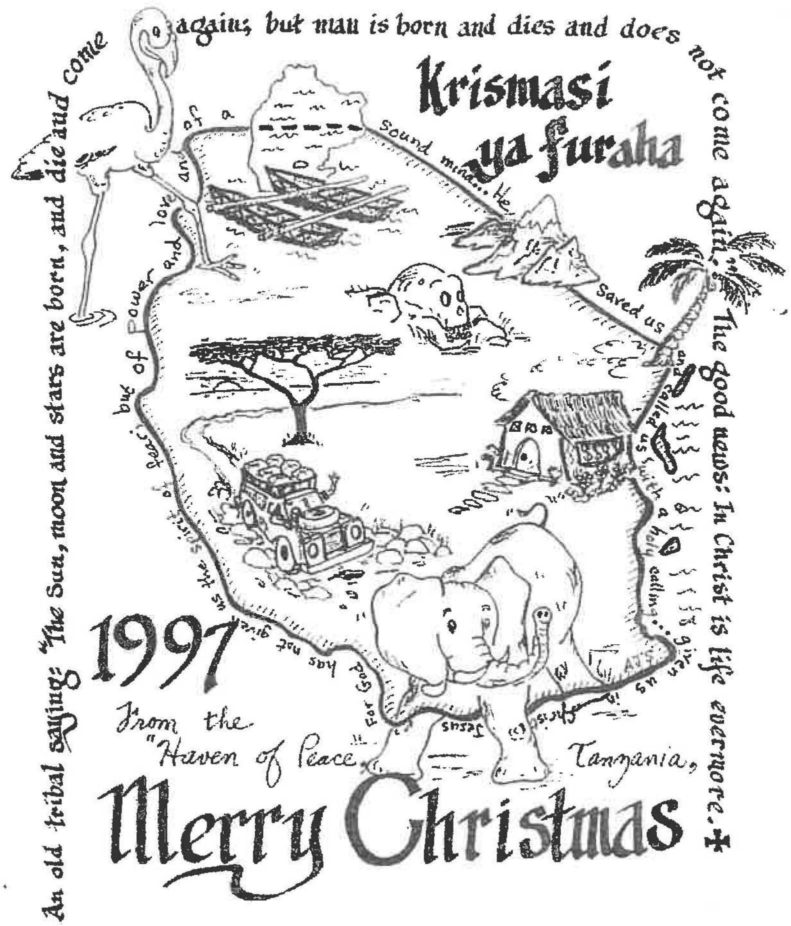

Figure 1. Tanzania.

Rather than telling about ... learning Swahili and bartering for fruit, fish, and anything else in the market places; a five day hike up Kilimanjaro in white-out conditions; viewing game on the Serengeti in a broken-down Land Rover; diving with dolphins and sea turtles in the Indian Ocean; boiling drinking water daily and sleeping under mosquito netting and repairing screening and dozens of other routine activities which become adventures when the water supply is cut off to the city, when power is rationed to two days a week for each section of the city, and when one has to communicate to shop keepers in primitive Swahili and pencil drawings, ... let me briefly discuss the university system, the students, and teaching conditions.

Primary public education is in theory available to all Tanzanian children. Walking is the only means of transport in most rural communities; so mere distance precludes schooling for many. Due to a number of reasons which would be an entire essay in itself, the government often reneges on programs it has budgeted. In particular, many teachers work for months without being officially paid. Hard-pressed to support their own families, some of these teachers charge daily tuition for their services; thus if a student arrives at the classroom without the proper fee, that child sits under a tree for the rest of the day without instruction. Five percent of this pool of primary children are successful enough to proceed on to secondary school, and five percent of that pool will proceed on to college. Provided the student has done sufficiently well on their exit exams from secondary school and provided they are within the quota of students allocated from each of the twentyone provinces of Tanzania, the student will be invited to apply to the university. In their application, the student selects and ranks three areas of study available at the university. From this pool, the faculty choose a quota of students for their respective departments. Every slot is filled; and once a student starts a major field of study there is no opportunity to change; once a math major, always a math major even if one later discovers that one's talents lie elsewhere. Acceptance into the university translates into a small government stipend and a slim opportunity to eventually secure a well-paying job. Since unemployment in this country runs at about 80% and since the typical wage for menial labor is less than $1 per day, and even though they know that no jobs await them upon graduation, the students are proud to be at the university. When the government is unable to pay their meagre stipends, the whole student body strikes; bold, strident speeches fill the amphitheatres and no class meets until the government "finds" some money. Once I tried to give an optional quiz in a class during a strike day, and a minor riot broke out. No less than six strikes, lasting anywhere from a few hours to several days, took place during my year at the University. Another time the government unilaterally cancelled classes for two weeks, apparently because the money had really vanished, at least from the ken of the official bursars.

The 5000 strong student body at the University of Dar es Salaam is typical of most student bodies; one third of the students are serious students, another one third are passable, and another one third somewhat clueless. The students tend to function as groups. That is, large numbers of students will have exactly the same programs and classes all during their three years at the university. Spokespeople and leaders emerge. Communal test-taking is commonplace; what we would call cheating has no stigma, for if one student arrives at an answer, that answer is transmitted in almost magical fashion throughout the class during exams. Despite exhortations that each person is responsible for their own work and despite appropriating exam papers from the more extravagant offenders, I had limited success in curtailing this habit. For example, on one class quiz, the correct short-answer-response to a group-theoretic problem was the number 17; many 19's and 7's appeared as answers, not because the students had miscalculated, but because the students had either misread 7 as 9 or failed to see the digit 1 on another student's paper.

Tanzanian students are exceedingly respectful; during a lecture no audible shuffling of books or papers or movement of chairs occurs, even though the classrooms are packed. During my first term, my linear algebra class had 130 students and my abstract algebra class had 85. Contrast this with North America where surely no abstract algebra class exceeds 30. In fact, during the first month of the term my abstract algebra class met in a room which could physically hold only 30 students; latecomers had to peer through windows or congregate in the hallway craning necks and later borrowing lecture notes. No other classrooms were available. Space is at a premium: too many students for the facilities. The buildings date back to a kinder era, when grant money and high prospects associated with the Cold War flowed more freely. Making up class times before breakfast or during lunch is universally anathema. Sunlight is from 6 am to 6 pm; and my classes were scheduled right up to 6 pm on Friday evening. After the sun sets, the lecture halls grow dark, for most bulbs have long since burnt out and no replacements are available. Power outages are frequent. On one late afternoon of power failure shortly before exams, we convened at an outside cement patio with sloping hills on all sides forming a viewing arena; I gave a lecture, scribbling with chalk on the coarse cement, stopping only when my chalk supply was dust. Books are in short supply; departmental libraries have but 15 copies or so of texts for each class; these I parcelled out to small student groups, appointing a team leader to be responsible for each book. Even had books been available, few students could purchase them on their stipend, for a book is comparable to two months of wages. Although English is an official language of Tanzania, few of the science students are comfortable speaking English publicly before so many; so during tutorial sessions students pass notes to the front requesting various exercises to be worked. Oral questions when volunteered are nearly inaudible because a low voice in addressing an elder is a traditional sign of respect; so I'd ask for the question to be repeated while moving closer to the speaker. When I'd make a mistake on the small, discolored and pitted blackboard, the writing on which is barely discernible beyond the first few rows of desks, at least one student would invariably notice and rap a pencil tip in staccato fashion on the desk top, so catching my attention. In Botswana classrooms, the corresponding custom is finger snapping.

Almost all lecturers at the University are both Tanzanians and alumni of the University of Dar es Salaam, the primary reason being monetary. No expatriate can afford to work at the lectureship wages of about $100 per month, albeit supplemented by modest housing, utility, and servant stipends. Hence many of the professors are entrepreneurs, orchestrating several ventures in order to provide for themselves and their extended families. As the new faculty member in the department, I was appointed seminar leader; although a few folks volunteered when asked, it became evident that the staff were really too busy to prepare yet another talk. Speculating that affairs might be similar at the universities of neighboring countries, I offered to give some math talks at these countries and wound up visiting and speaking at the Universities in Lusaka, Zambia, and Harare, Zimbabwe, and Gaborone, Botswana. In contrast to the University in Tanzania, the faculty at these institutions are from all over the world: many Eastern Europeans, a few Brits, some Indians, and an assortment of African nationals many of whom have been educated in the U.S. One striking vignette, which both displays the strengths and the weaknesses of the typical Sub-Saharan African country (not including South Africa): when I asked to view the math library holdings at the University of Zimbabwe, my hosts showed me a large light-filled room of many windows with tier after tier of bookshelves—almost all of which stood empty. In general, a tremendous amount of aid money has been sent and continues to be sent from many countries to bolster the infrastructure of Sub-Saharan Africa. Despite deep-rooted corruption throughout the fabric of African society, the good news about higher education is that even in the poorest of these countries, the university although handicapped by a checkered resource supply is in good shape, especially when contrasted with the plight of general education, transportation, health conditions and water development in those countries.

Trochoids.

Perhaps the most beautiful of curves studied in calculus are the *trochoid* family. In the mathematical zoo of the typical calculus text, these trochoids are presented as the curves traced out by a tack embedded somewhere in a wheel, which in turn is rolling around a circle. In figure 2a, the bold *rail-road wheel* has a tack in its rim which overlaps the thin-lined circle "road"; the dashed curve is the trochoid, the path traced out by the tack. In figure 2b, the wheel is rotating on the inside of the circle. In this paper, we wish to present these trochoids in a more *wild* yet natural setting where the trochoids are both hidden and defined by a family of lines. This envelope perspective will contrast favorably with the traditional rolling wheel perspective.

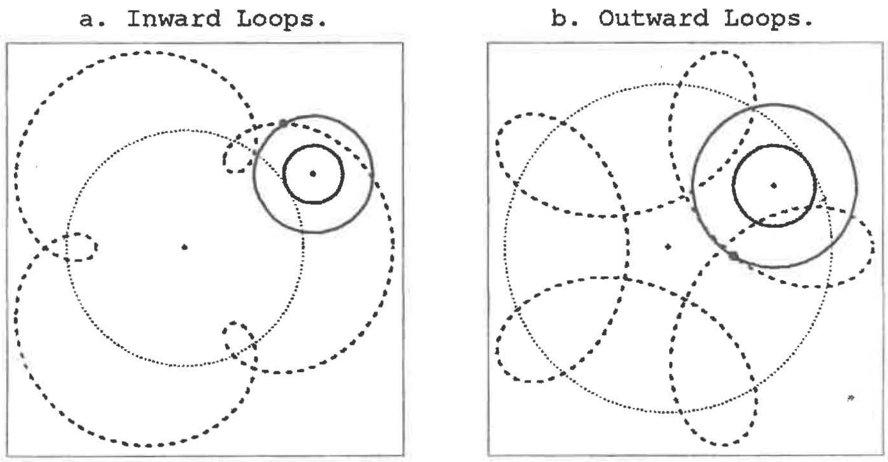

Figure 2. The Trochoids.

To present the mathematical *bush* in which we hope to spot the trochoids, consider the following problem: Two people A and B walk along holding a taut bungee chord between them. We shall assume that this bungee chord stretches uniformly and can collapse to a point. This family of chord positions defines another curve, called the envelope. See figure 3, wherein A proceeds along a line segment from one end to the other while B circles an elliptical track three times; the envelope is then a kind of three pronged tiara.

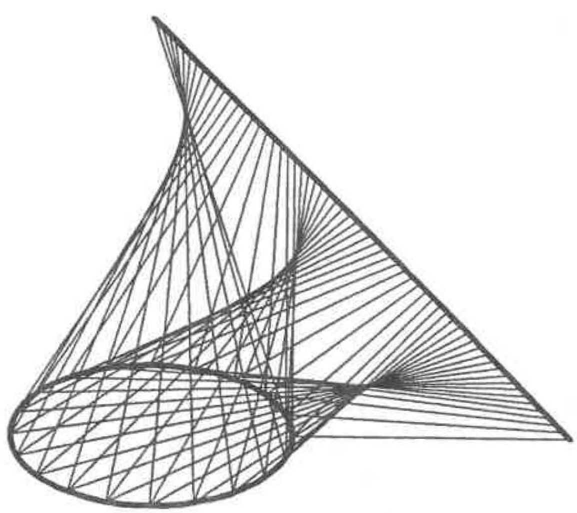

Figure 3. A tiara envelope.

From the standpoint of calculus, these families of lines are tangent lines to an unknown curve, and a good deal of differential equation analysis of previous centuries used this approach. Such an approach has great geometric appeal at the once ruinous cost of having to generate tedious and almost unwieldly drawings and calculations. But mathematical software can at times make this cost affordable.

In particular, consider two people A and B walking uniformly counterclockwise around the unit circle holding a taut bungee chord between them, where A makes p circuits in the time that B makes q circuits. The envelope of these bungee chord lines this time is an *epicycloid*, a special case of the trochoid of figure 2(a) where the tack is on the rim of a nonoverlapping wheel. When the two walkers proceed in opposite directions the envelope is a *hypocyloid*, a special case of the trochoid of figure 2(b) where the tack is again on the rim of a nonoverlapping wheel; the resulting envelope develops outside the circle in this case.

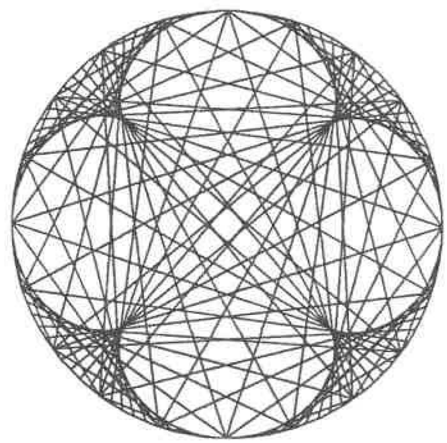

Figure 4. An epicycloid envelope with $p = 5$, $q = 1$.

The classic way to characterize the envelope in the x-y plane as an equation is to introduce a parameterizing variable t, writing the lines as the zero level set of a function $F(x, y, t)$, and then to solve the system $\{F(x, y, t) = 0, F_t(x, y, t) = 0\}$ where F_t is the partial derivative of F with respect to t; the resulting curve is the so-called *fold set*, where the graph of $F = 0$ would fold upon itself if projected onto the x-y plane. See [**2**, p. 76] for more details. In this problem, the family of chords is $[(\cos pt, \sin pt), (\cos qt, \sin qt)]$ so that

$$F(x, y, t) = x(\sin pt - \sin qt) + y(\cos qt - \cos pt) + \sin(qt - pt). \tag{1}$$

The details of capturing the epicycloid equations from this approach appear in [**7**].

To represent these kinds of curves, and in fact any kind of curve, in terms of an envelope we pose the following problem:

A Bungee Chord Characterization: Given a curve Γ find a function pair (f, g) such that the envelope of the lines with endpoints $(\cos f(t), \sin f(t))$ and $(\cos g(t), \sin g(t))$ is Γ.

So for example, for the epicycloid and hypocycloid classes, the function pairs are the linear functions (pt, qt) where p and q are different nonzero integers. In general, if A and B proceed at varying rates about the unit circle then we have the following result where S_h and C_h represent $\sin h(t)$ and $\cos h(t)$, respectively, where h is a function in terms of the variable t.

Theorem: A Parameterized Envelope. The envelope formed by the family of chords $[(\cos f(t), \sin f(t)), (\cos g(t), \sin g(t))]$ where f and g are different differentiable functions is the parametric set of equations:

$$(x(t), y(t)) = (\frac{f'C_g + g'C_f}{f' + g'}, \frac{f'S_g + g'S_f}{f' + g'}). \tag{2}$$

To derive these equations note that a point (x, y) on any of these lines must satisfy the equation

$$y - \sin f(t) = \frac{\sin g(t) - \sin f(t)}{\cos g(t) - \cos f(t)}(x - \cos f(t))$$

for some t. Rearranging terms, we can represent the set of points (x, y, t) satisfying such equations as the zero level set of the function $F(x, y, t) = x(S_f - S_g) + y(C_g - C_f) + S_{g-f}$. Hence $F_t = x(C_f f' - C_g g') + y(S_f f' - S_g g') + C_{g-f}(g' - f')$. After a good deal of straightforward simplification the solution set to the equations $\{F = 0, F_t = 0\}$ can be written in the desired form.

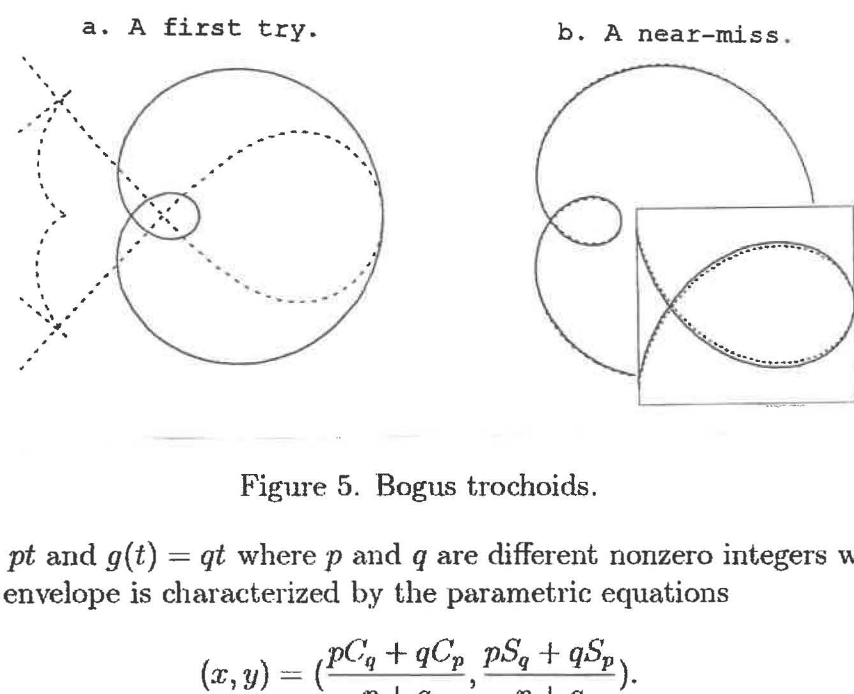

Figure 5. Bogus trochoids.

Thus if, $f(t) = pt$ and $g(t) = qt$ where p and q are different nonzero integers with $p \neq -q$, then the corresponding envelope is characterized by the parametric equations

$$(x, y) = (\frac{pC_q + qC_p}{p + q}, \frac{pS_q + qS_p}{p + q}). \tag{3}$$

The reader can verify that these equations are equivalent to the standard equations for the epicycloid and hypocycloid where the circle radius of the road is $\frac{p-q}{p+q}$ and the circle radius of the wheel is $\frac{|q|}{p+q}$ where $p > |q|$.

In terms of the bungee chord characterization problem, we would like to find that function pair (f, g) which yields a given trochoid curve Γ. Since these trochoid curves are so strongly related to the epicycloid and hypocycloid, one might guess that finding the bungee chord characterizations for the trochoids ought to be easy. A poor first guess to match a translate Γ of the polar curve $r = \frac{1}{2} + \cos\theta$ appears in figure 5(a) where $f(t) = 2t + 2\sin 2t$ and $g(t) = t$; Γ is the solid curve and the envelope is the dashed curve. One "near miss" for the bungee chord characterization of Γ is the function pair $(f, g) = (2t + \frac{\pi}{2}\sin t, 2t - \frac{\pi}{2}\sin t)$; but as one can see from figure 5(b) (where the inset is an enlargement of the inner loop) or from some curve analysis, when the curve corresponding to this function pair is compared against the corresponding *true* trochoid, the bogus curve is revealed. At this point, finding the bungee chord characterization for the trochoids remains an open problem.

However, some genuine trochoids can be flushed from their hiding places by employing the higher powered partial derivatives of F of (1). Let $\mathcal{F}_u$ represent the projection of the curve of intersection between the surfaces $F = 0$ and $\frac{\partial^u F}{\partial t^u} = 0$ where u is any positive integer. Figure 6a gives an idea of the "twistedness" of the surface of $F = 0$ for the function pair $(f, g) = (pt, qt)$ where in this case $p = 2$ and $q = 1$; and figure 6b gives a close-up with the paths as given by $\mathcal{F}_1$ and $\mathcal{F}_2$ in solid and dashed curves, respectively. Projections of these curves onto the x-y plane suggest some natural conjectures. That is, from figure 7a(2), the curve $\mathcal{F}_2$ looks like a trochoid; and sifting through the equations yields the following result where we use the word "harmonic" in the sense of obtaining similar patterns at "deeper" levels.

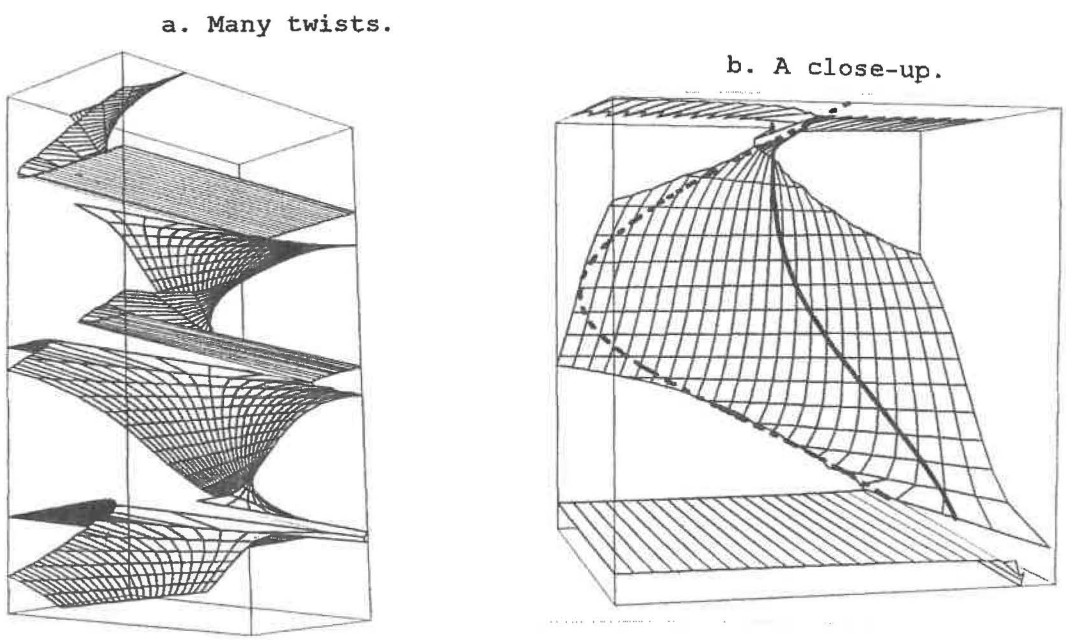

Figure 6. The Manifold $F(x, y, t) = 0$.

Theorem 2. Harmonic Envelopes. Let n be a positive integer. Let F be the function in (1). The curve $\mathcal{F}_{2n}$ is a trochoid whose parametric representation is

$$(x, y) = \left(\frac{(p^{2n} - (q-p)^{2n})C_p + ((q-p)^{2n} - q^{2n})C_q}{p^{2n} - q^{2n}}, \frac{(p^{2n} - (q-p)^{2n})S_p + ((q-p)^{2n} - q^{2n})S_q}{p^{2n} - q^{2n}}\right). \quad (4)$$

To generate (4) note that $\frac{\partial^{2n} F}{\partial t^{2n}} = 0$ is equivalent to the following:

$$x(p^{2n}S_p - q^{2n}S_q) + y(q^{2n}C_q - p^{2n}C_p) + (q-p)^{2n}S_{q-p} = 0.$$

From $F=0$, we have $x = \frac{S_{p-q}+y(C_p-C_q)}{S_p-S_q}$. Substituting this expression into $F_t = 0$ along with much straightforward simplication, we have the desired equation for y. The formula for x is similarly derived. The reader can verify that any parametric set of equations of the form

$$(x, y) = (\alpha\cos(mt) + \beta\cos(kt), \alpha\sin(mt) + \beta\sin(kt)), \tag{5}$$

where α and β are any real numbers and m and k are rational numbers, will trace out a trochoid as t increases, with the one exception being when x or y is the zero function, corresponding to the curve being a line segment.

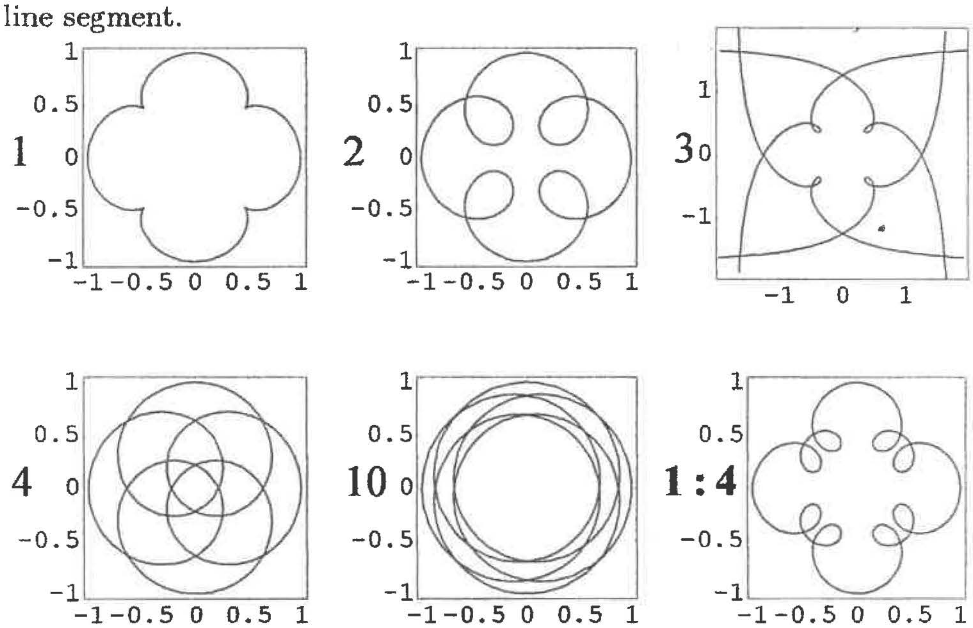

Figure 7. Partial Derivative Envelopes for $p=5$, $q=1$.

Figures 7a(4) and 7a(10) are the trochoids $\mathcal{F}_4$ and $\mathcal{F}_{10}$. The odd orders fail to yield trochoids however; for example figure 7a(3) gives $\mathcal{F}_3$. Experimenting with other combinations give interesting curves as well; for example, figure 7a(1:4) is an exotic curve which is the intersection of the surfaces $F_t = 0$ and $F_{tttt} = 0$.

To better understand this phenomenon as to why the even orders yield trochoids and the odd orders greater than one fail to do so, it is useful to consider the fractional derivative and look at the curve $\mathcal{F}_u$ defined as the x-y projection of the solution set of $\{F=0, \frac{\partial^u F}{\partial t^u} = 0\}$ where u is any real number. In particular, with this operator it is helpful to see how the epicycloid $\mathcal{F}_1$ in figure 7a(1) *morphs* into the trochoid $\mathcal{F}_2$ of figure 7a(2) as u increases from 1 to 2. As will be shown the derivative metamorphosis of $\mathcal{F}_1$ becoming $\mathcal{F}_2$ encounters a stage $\mathcal{F}_{1.35}$ which is similar in kind to the stage $\mathcal{F}_3$ in the metamorphosis of $\mathcal{F}_2$ into $\mathcal{F}_4$.

The fractional derivative (see [5] and [6]) dates back to 1867, and is now somewhat out of vogue, simply because it has been supplanted by more manipulable operators. However, the computer can unlimber much of this unwieldliness. Integral to its definition is the *gamma* function: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$, where $\alpha > 0$; extend this function to all real numbers by using the recursion relation

$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}$ for all α. See figure 8(a). Then the somewhat intimidating definition due to Grünwald (1867) of the $u^{\underline{th}}$ *fractional derivative* expanded about a where u and a are any real numbers is

$$\frac{d^u f(x)}{[d(x-a)]^u} = \lim_{N\to\infty} \{ \frac{[\frac{x-a}{N}]^{-u}}{\Gamma(-u)} \sum_{j=0}^{N-1} \frac{\Gamma(j-u)}{\Gamma(j+1)} f(x - j\frac{x-a}{N})\}. \tag{6}$$

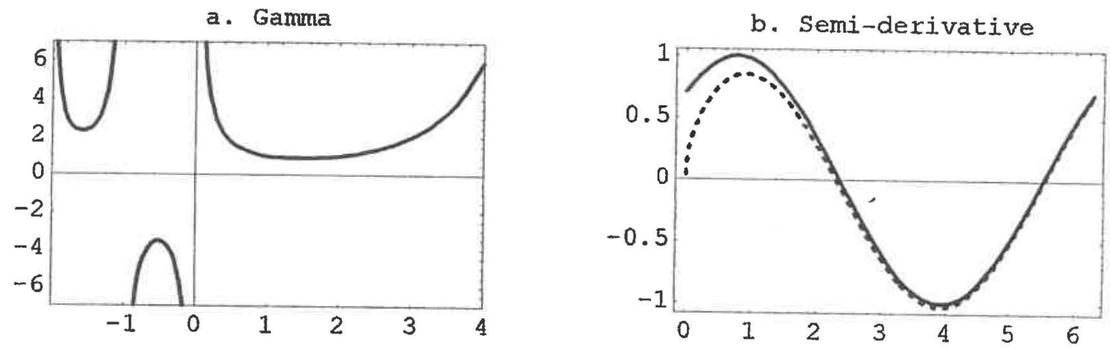

Figure 8. The Gamma function and the semi-derivative of the Sine function.

With a few appropriate tricks, we can find $\mathcal{F}_u$. First of all, we take $a = 0$. Two familiar properties of the ordinary derivative operator remain, namely, for any functions f and g and constants α and β we have

•Linearity: $$\frac{d^u(\alpha f(x) + \beta g(x))}{dx^u} = \alpha \frac{d^u f(x)}{dx^u} + \beta \frac{d^u g(x)}{dx^u}. \tag{7}$$

•A Limited Chain Rule: $$\frac{d^u f(\alpha x)}{dx^u} = \alpha^u \frac{d^u f(\alpha x)}{d(\alpha x)^u}. \tag{8}$$

Some properties are hopelessly absent. In particular, it turns out that the $u^{\underline{th}}$ derivative of a periodic function fails in general to be periodic. To demonstrate this phenomenon, let's consider the *semi-derivative*, a special case of the general definition as presented by Courant and Hilbert [4,p.50]; applying their definition to the sine function gives

$$\frac{d^{\frac{1}{2}} \sin x}{dx^{\frac{1}{2}}} = \sqrt{2} \int_0^{\sqrt{\frac{2x}{\pi}}} \cos(x - \frac{\pi}{2}t^2)dt. \tag{9}$$

Fresnel integrals of this kind are found in the standard reference [1, p. 300]. The dashed curve of figure 8(b) depicts the graph of the semiderivative of $\sin x$ on $[0, 2\pi]$. As x increases, the semi-derivative of $\sin x$ converges to $\sin(x + \frac{\pi}{4})$, the solid curve of 8(b). And in general for large x it can be shown [5, p. 110] that

$$\frac{d^u \sin x}{dx^u} = \sin(x + \frac{\pi}{2}u) + \frac{x^{-1-u}}{\Gamma(-u)} - \frac{x^{-3-u}}{\Gamma(-u-2)} + \dots$$

and

$$\frac{d^u \cos x}{dx^u} = \cos(x + \frac{\pi}{2}u) + \frac{x^{-2-u}}{\Gamma(-u-1)} - \frac{x^{-4-u}}{\Gamma(-u-3)} + \dots.$$

For $u > 0$ and for x sufficiently large we use the approximate formula:

$$\frac{d^u \sin x}{dx^u} \approx \sin(x + \frac{\pi}{2}u) \quad \text{and} \quad \frac{d^u \cos x}{dx^u} \approx \cos(x + \frac{\pi}{2}u). \tag{10}$$

Thus $\frac{\partial^u F}{\partial t^u} = 0$ is asymptotically equal to

$$x(p^u \sin(pt+\frac{\pi u}{2})-q^u \sin(qt+\frac{\pi u}{2}))+y(q^u \cos(qt+\frac{\pi u}{2})-p^u \cos(pt+\frac{\pi u}{2}))+(q-p)^u \sin((q-p)t+\frac{\pi u}{2}) = 0. \tag{11}$$

Hence we can sketch $\mathcal{F}_u$ for any u. The portions of the paths in $\mathcal{R}^3$ corresponding to the x-y planar curves $\mathcal{F}_u$ "live" high up on the surface $F = 0$. But their projections will be asymptotically periodic as depicted in the following two figures where each diagram is labelled by its corresponding u value.

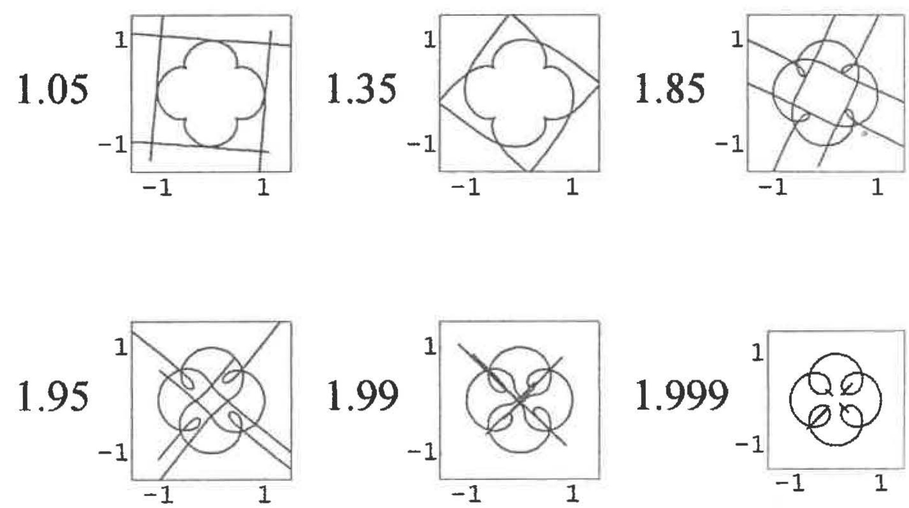

Figure 9. An epicycloid morphing into a trochoid.

Figure 9 shows how the epicycloid of figure 7(1) changes with the fractional derivative into the trochoid of figure 7(2). To describe this transformation, we can imagine a surgical team of four simultaneosly snipping figure 7(1) at the four points furthest from the origin, so creating eight thrashing strands which explode outwards to the point at infinity somewhat like the severed ends of a piano string erupting under great tension; each surgeon grapples with their severed two strands, wrestling their patient clockwise, causing the curve to cross at its cusps so forming loops, as they push the strands back into themselves, and ultimately melding the ends at the four points nearest the origin. Figure 10 shows how the trochoid emerges from "nothing" into an epicycloid by examining the fractional derivative homotopy as u ranges from 0 to 1.

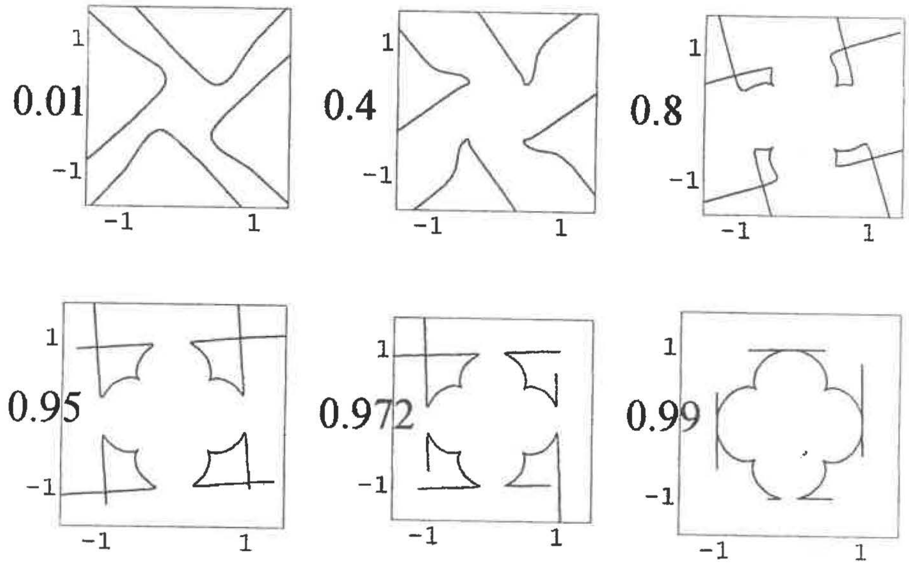

Figure 10. An epicycloid emerging from nothing.

A smoother homotopy however is obtained by imagining that A and B travel about separate tracks, one above the other; this time the family of chords traces out a two-dimensional surface in $\mathcal{R}^3$, and each horizontal cross section for the bungee-chord-characterization pair $(f, g) = (pt, qt)$ turns out to be a trochoid. That is, a two-dimensional parameterization of this surface is

$$G(t, \lambda) = (\lambda \cos pt + (1 - \lambda) \cos qt, \lambda \sin pt + (1 - \lambda) \sin qt, \lambda) \tag{12}$$

where $0 \le t \le 2\pi$ and and λ is a real number. Note that $G(t, 1)$ and $G(t, 0)$ are the two unit circle tracks on which A and B are walking. See figure 11. That is, by (5), for any real λ, the horizontal cross sections of this surface are the trochoids

$$T(p, q, \lambda) = (\lambda \cos pt + (1 - \lambda) \cos qt, \lambda \sin pt + (1 - \lambda) \sin qt). \tag{13}$$

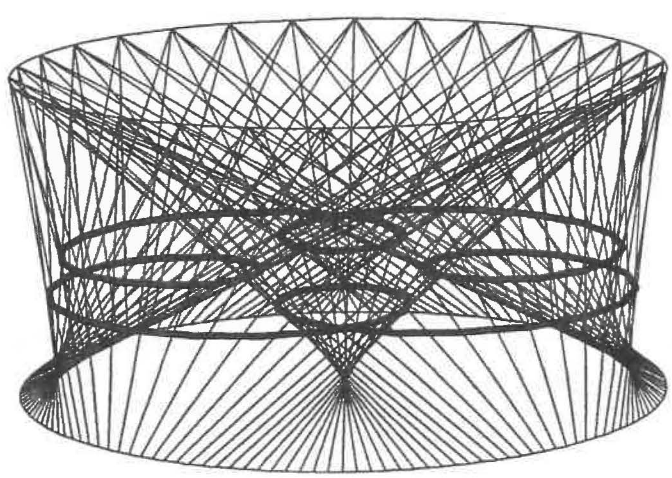

Figure 11. Layers of trochoids.

That is, an alternate trochoid model to the rolling wheel model is the following.

The Alternate Model: The Trochoid as a Tack on a Bungee Chord. The trochoid can be defined as the curves traced out by a tack in a bungee chord whose ends are held by two runners proceeding around a circular track at rational relative speeds.

To encompass all cases, we envision the two runners as each holding onto a point of a cosmic bungee line which stretches linearly as the distance between the two runners varies. In this way, the tack can be positioned anywhere along the bungee line, not just between the two runners.

The reader may enjoy verifying the following:

- the polar flowers $r = \cos(\frac{m}{n}\theta)$ where m and n are integers are the trochoids $T(n+m, n-m, \frac{1}{2})$;
- the polar cardioid $r = 1 + a\cos\theta$ where a is any positive number is the translated trochoid $\frac{a+2}{2}T(1, 2, \frac{a}{a+2}) + (\frac{a}{2}, 0)$;
- the epicycloid and hypocycloid of p cusps are $T(p+1, 1, \frac{1}{p+2})$ for $p \geq 0$ and $T(p-1, -1, \frac{1}{2-p})$ for $p \geq 3$, respectively;
- some trochoids of p non-intersecting "inner" and "outer" loops are $T(p+1, 1, \frac{1+2p}{p^2+2p})$ for $p \geq 2$ and $T(p-1, -1, \frac{1-2p}{p^2-2p})$ for $p \geq 3$, respectively.

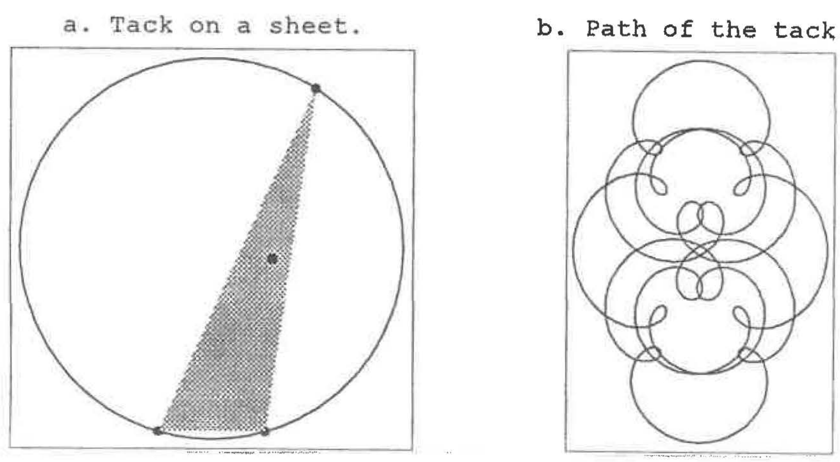

Figure 12. Generating general trochoids.

Just as the wheel model of the trochoid generalizes to multiple wheels in [**3**], so the bungee chord model generalizes. That is, via a *barycentric coordinates* idea (see [**4**,p. 19], for example), if $\sum_{i=1}^{n} \lambda_i = 1$ and a_i are rational numbers then any curve of the form

$$(\sum_{i=1}^{n} \lambda_i \cos(a_i t), \sum_{i=1}^{n} \lambda_i \sin(a_i t))$$

is a generalized trochoid except when one component is identically zero. (One can avoid the degenerate case of generating a line segment by specifying that the corresponding Fourier functions $\{\cos a_i t\}_{i=1}^{n} \cup \{\sin a_i t\}_{i=1}^{n}$ is a linearly independent set.) Such a generalized trochoid is generated by following the path of a tack in a stretchable rubber sheet whose corners are held by n people where person-i moves around the unit circle with velocity $a_i t$. Actually, we view the entire x-y plane as a cosmic rubber sheet which can fold in upon itself, wherein the tack moves around in the plane as a fixed linear combination of the position of the runners. Figure 12 depicts the path of a tack in a triangular sheet whose corners are positioned at $(\cos t, \sin t)$, $(\cos 5t, \sin 5t)$ and $(\cos 17t, \sin 17t)$ and $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$ and $\lambda_3 = \frac{1}{6}$, respectively.

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