

Teaching Complex Analysis as a Lab-Type Course with a Focus on Geometric Interpretations using *Mathematica*

William M. Kinney

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Abstract

I taught complex analysis for the first time in my career during the spring of 2013. I decided to do something “radical” and teach it as a lab-type course with a focus on geometric interpretations using the computer program *Mathematica*. The students and I met in a computer lab and, during most meetings, we spent a large portion of our time experimenting and exploring using *Mathematica* to visualize key concepts in complex analysis. Because of this, there was a heavy emphasis on viewing analytic functions as conformal mappings as well as considering associated vector fields and flows. *Mathematica* was used to make the concepts “come alive” through its animation capabilities. A description of some of these animations will be the main focus of this paper. I will also briefly discuss how I helped students learn more basic content through the use of many 10-minute video lectures (I also taught basic *Mathematica* code in these video lectures).

1 Introduction

The idea of a “flipped” or “inverted” classroom has become trendy in recent years at all levels of education [1] (<http://teconomy.com/2012/08/flipped-classrooms-turn-learning-on-its-head/>). As of early June in 2013, *Wikipedia* [8] describes it as

“Flip teaching (or flipped classroom) is a form of blended learning which encompasses any use of technology to leverage the learning in a classroom, so a teacher can spend more time interacting with students instead of lecturing. This is most commonly being done using teacher-created videos that students view outside of class time. It is also known as backwards classroom, reverse instruction, flipping the classroom, and reverse teaching.” http://en.wikipedia.org/wiki/Flip_teaching

It is a long-term goal of mine to ultimately make all the classes I teach into some version of a flipped classroom, but it does take a lot of work to get there. Pedagogically-speaking, this is a vision I want to implement in my courses based on the opinions I have listed below.

1. Students are more engaged during class, especially if they are held personally accountable for the activities they do in class. Accountability for this engagement can take place during class or after class. If they are available, teaching assistants can be valuable resources to help keep students accountable through assessment of student products related to these activities.
2. The classroom should be, and can be, the best place to gain exploration skills, strengthen conceptual understanding, learn how to ask good questions, and improve problem-solving skills because of direct instructor guidance. It can also be the best place to gain experience with problems that are deeper and more “authentic”. In fact, the classroom is an ideal place to collaboratively solve problems that might take a week or even a month to solve together. The instructor can guide this problem-solving teamwork by giving the students activities that will hopefully lead them to a solution. This also provides the instructor with opportunities to hone his or her own skills in identifying good problems and in working with the students to solve such problems.
3. Basic skills can be practiced in a flipped classroom, but it is a better use of class time for them to be practiced outside of the classroom. There are also many online resources to help students improve their basic skills. Students should still be assessed on their basic skills; perhaps through gateway exams and probably on regular exams as well.
4. Hopefully, the end result of all this will be better learning and better preparation for future learning and jobs. At the most basic level, there should be more learning that occurs just because of more classroom engagement happening. The students will hopefully develop a deeper desire to learn because of the confidence gained, because of the focus on exploration, and because of the connections made to real-life through significant and authentic problems.
5. As a Christian, I believe that the creation of a better learning environment in the mathematics classroom is a big part of God’s call on my life. I want to glorify Him through the study and teaching of mathematics and to help people find their own calling, develop their own skills, and make the world a better place. I also think this environment will help me develop better personal mentoring relationships with more of my students because it will facilitate more time for one-on-one interaction.
6. The creation of online materials in both written and video format can help an instructor and an institution to broaden their audiences. For instance, instructors can create blogs or *YouTube* channels that, if done well, can become relatively popular around the world. I believe this should be considered a valid form of professional activity and development at many institutions and it also can be thought of as a calling from God to make the world a better place and to glorify Him.
7. All residential colleges and universities may be economically forced towards some form of this model for many of their courses. In order to differentiate ourselves from purely online instruction and provide value commensurate with our tuitions, we need to become more engaged with our students in the deeper issues and problems related to our subjects. Sufficiently motivated and disciplined students can learn basic skills and factual knowledge on their own online. Why should they go to college unless we offer them something deeper? Unmotivated and undisciplined students will benefit from a more “hands-on” and relationship-focused approach as well.

During the spring of 2013, I taught complex analysis at Bethel University for the first time. I had plans to implement this vision for the reasons mentioned and I had a small amount of success doing so the first couple weeks. I was, however, thwarted in fully realizing complex analysis as a flipped classroom for the whole semester because my computer broke down a couple weeks into the course. By the time it was fixed a week and a half later, I was too far behind to catch up, both in terms of creating *Mathematica*-based computer activities for the course and in terms of making instructional videos. In order to survive during the intense busyness of the semester, I reverted back to my standard mode of teaching. In this standard mode of teaching, I still made extensive use of *Mathematica* with my students, just not in an activity-oriented framework. In this rest of this paper, I will describe how I made use of *Mathematica*. The main focus will be on how I used *Mathematica* to illustrate fundamental concepts in complex analysis, but I will begin with a description of a few of its capabilities for more basic courses.

2 Using *Mathematica* as a Tool to Implement a Flipped Classroom in Basic Courses

Mathematica is a very powerful software application. It was known mostly for its symbolic computation capabilities during the first fifteen to twenty years of its development, though in the past five to ten years its visualization, numerical computation, data-access, and data-manipulation capabilities have been greatly improved. I have been making use of it extensively in my courses for about fifteen years. For the most part, I have used it as a calculation-checking and visualization tool for the purpose of illustrating concepts to my students. But I have also used it fruitfully in my own research in ordinary differential equations [2,3] and actuarial mathematics (done as a student/faculty summer research team with Jacob Smith) [4]. I have also required my students to use it to do real-world (“authentic”) projects, especially in calculus-sequence courses.

One key tool for realizing *Mathematica*’s visualization capabilities is the command `Manipulate`. In fact, `Manipulate` can be used to make many kinds of interactive output, including interactive numerical output. For instance, the line of code `Manipulate[Table[Prime[n],{n,1,m}],{m,1,10,1}]` will make a slider-enabled output box (shown in Figure 1) that will show the first m primes for different integer values of m from 1 to 10 (“`Prime[n]`” returns the n^{th} prime and “`Table`” creates the list of primes from the first prime through the m^{th} prime. Figure 1 shows the animation “frozen” at $m = 8$.

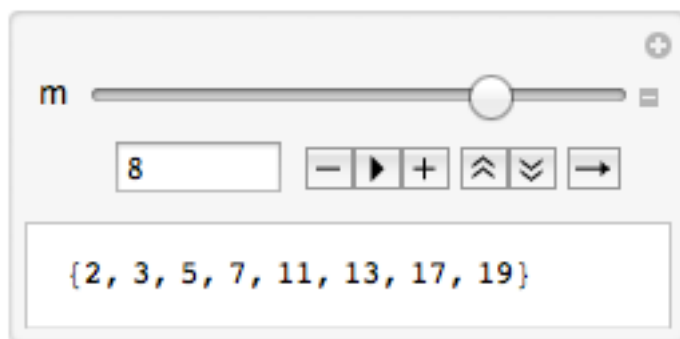


Figure 1: Snapshot of `Manipulate` animation to show the first m primes when $m = 8$.

As another example, with the help of `Binomial[n,k]`, which returns the binomial coefficient $\binom{n}{k}$, we can quickly create an interactive version of Pascal's Triangle (see a snapshot in Figure 2 with $m = 9$) with the line:

```
Manipulate[Column[Table[Binomial[n,k],{n,0,m},{k,0,n}],Center],{m,0,10,1}]
```

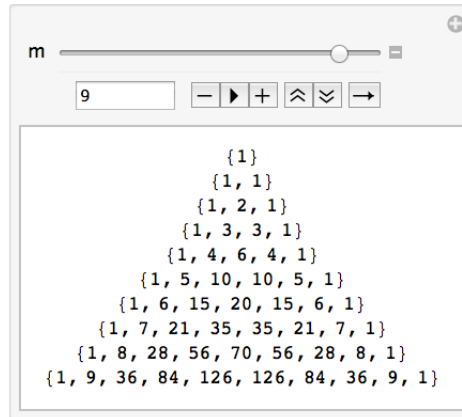


Figure 2: Snapshot of Manipulate animation to show Pascal's Triangle with $m + 1 = 10$ lines.

In a first-semester calculus class, one of the most fruitful uses of Manipulate is to help the students visualize the concept of local linearity. In Figure 3 we see a snapshot of an animation used to illustrate this concept as we zoom in on the graph of $f(x) = x^2$ near the point $(1, 1)$. The animation parameter ϵ is used to determine the size of the “zoom box” (it's the “radius” of the box relative to the maximum norm $\|\cdot\|_\infty$).

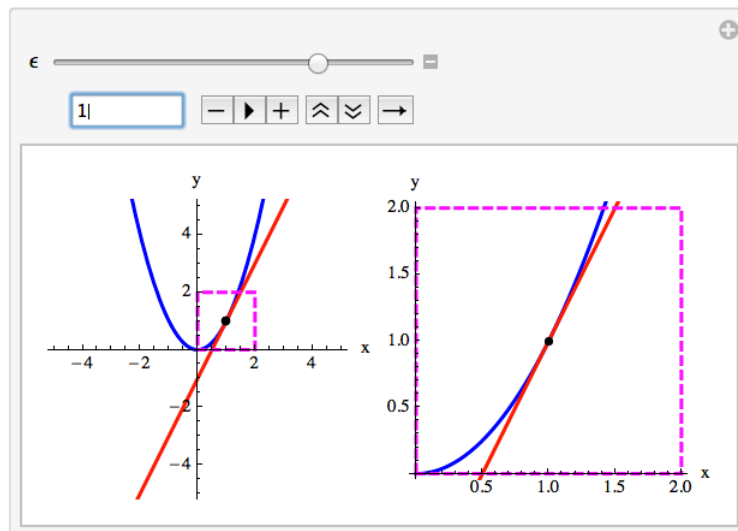


Figure 3: Snapshot of Manipulate animation to show the local linearity of $f(x) = x^2$ near $x = 1$.

```

x[t_] := Cos[3 t]; y[t_] := Sin[5 t]; r[t_] := {x[t], y[t]}; v[t_] := r'[t]; a[t_] := r''[t]; T[t_] :=  $\frac{v[t]}{\sqrt{v[t].v[t]}}$ ;
Nunit[t_] :=  $\frac{T'[t]}{\sqrt{T'[t].T'[t]}}$ ;  $\kappa[t_] := \frac{\text{Abs}[x'[t] y''[t] - y'[t] x''[t]]}{(x'[t]^2 + y'[t]^2)^{3/2}}$ ; R[t_] := 1/ $\kappa[t]$ ; speed[t_] := Sqrt[v[t].v[t]];
dist[t_] := NIntegrate[speed[t], {t, 0, t}];
Manipulate[
Grid[
{
Show[ParametricPlot[r[t], {t, 0, b}], PlotStyle -> {Thick, Blue}], ListPlot[{r[b]}, PlotStyle -> {Black, PointSize[.04]}],
Graphics[{Thick, Red, Arrow[{0, 0}, r[b]}]}, Graphics[{Thick, Magenta, Arrow[{r[b], r[b] + r'[b]/4}]}],
Graphics[{Thick, Darker[Green], Arrow[{r[b], r[b] + r''[b]/8}]}]}, Graphics[{Thick, Orange, Circle[r[b] + R[b] * Nunit[b], R[b]}]},
PlotRange -> {{-2, 2}, {-2, 2}}, AxesOrigin -> {0, 0}, AxesLabel -> {"x", "y"}, TicksStyle -> 15, ImageSize -> Medium,
Plot[x[t], {t, 0, b}], PlotStyle -> {Thick, Orange}, PlotRange -> {{0, 2  $\pi$ }, {0, 25}}, AxesLabel -> {"t", "curvature"}, ImageSize -> Medium]],
Plot[{speed[t], dist[t]}, {t, 0, b}], PlotStyle -> {{Thick, Magenta}, {Thick, Blue}}, PlotRange -> {{0, 2  $\pi$ }, {0, 25}},
AxesLabel -> {"t", "speed (magenta) & distance (blue)"}, ImageSize -> Medium],
Plot[R[t], {t, 0, b}], PlotStyle -> {Thick, Orange}, PlotRange -> {{0, 2  $\pi$ }, {0, 25}}, AxesLabel -> {"t", "radius curv"},
ImageSize -> Medium]]], {b, .0001, 2  $\pi$ }, LabelStyle -> Large]

```

Figure 4: Code for Manipulate animation of the parametric curve $\mathbf{r}(t) = (\cos(3t), \sin(5t))$ and associated quantities.

One of my favorite uses of *Mathematica* is to model motion using parametric curves and to animate the position, velocity, and acceleration vectors as well as the curvature, radius of curvature, and osculating circle. Figure 4 shows the (relatively small amount of) code used to create Figure 5. This animation uses b , the right endpoint of the interval $[0, b]$, as the animation parameter (the snapshot freezes the picture at $b = 3$). It plots the parametric curve $\mathbf{r}(t) = (\cos(3t), \sin(5t))$ over the interval $[0, b]$ along with the position vector, velocity vector (scaled down), acceleration vector (scaled down), and osculating circle. We also see graphs of the curvature function and radius of curvature function along the right as well as the distance traveled and speed functions at the lower left.

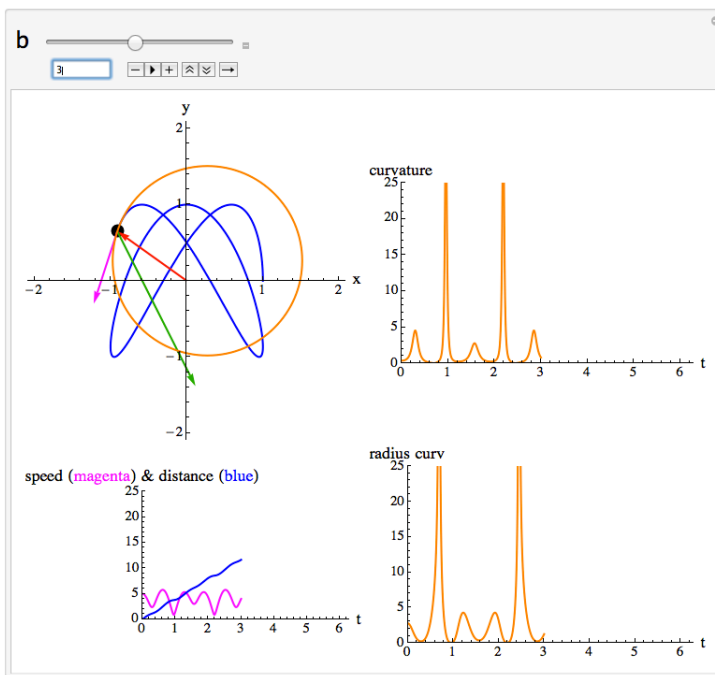


Figure 5: Snapshot of Manipulate animation of the parametric curve $\mathbf{r}(t) = (\cos(3t), \sin(5t))$ and associated quantities.

Conversion of my calculus-sequence courses to flipped classrooms will involve helping the students learn how to create these *Mathematica* animations for two purposes: 1) to aid their understanding and 2) to help them solve, or at least model, more complicated and authentic problems and situations. Instruction about the relevant *Mathematica* code will occur both in videos I make (where basic topic-oriented instruction also occurs) and in class through activities, individual help, and whole-class discussion and illustration. I have many ideas for week-long (or month-long) in-class problems. For instance, in Multi-variable Calculus (“Calculus 3”), I have a goal this fall (2013) to see whether or not we can be successful in using *Mathematica* to model some of the following kinds of motion (all of which will also require us to learn how to set-up and use *Mathematica*’s differential equation-solving capabilities): (1) the motion of a mass on a spring (harmonic motion), with and without friction, (and with and without external forcing) (2) the motion of a pendulum, with and without friction (and with and without external forcing), (3) projectile motion with and without air resistance in a fixed three-dimensional reference-frame, (4) accurately modeling the orbits of the first four planets around the sun, (5) projectile motion with and without air resistance in a rotating three-dimensional reference frame on the surface of a sphere (so the Coriolis effect must be taken into account), (6) launching a satellite into orbit, (7) devising a trip to the moon and back, (8) (piecewise) construction of a roller coaster and a car moving along the track in a physically-accurate way. I’m confident I can help the students be successful for (1), (2), (3), and (4). I’m less sure of my capabilities in helping them with the others. But we will try and see whether we succeed or not. I’m confident there will be joy in using the gifts God has given us in our attempts even if we do not achieve full success.

3 Using *Mathematica* as a Tool to Implement a Flipped Classroom in Complex Analysis

In complex analysis, one of the first fundamentally new concepts encountered is the geometric interpretation of complex number multiplication as resulting in the multiplication of moduli and the addition of arguments, modulo 2π . In Figure 6, we see a snapshot of an animation used to illustrate this concept. This time, however, the interactivity is not occurring through the use of a slider, but rather by clicking on each factor in the product (the red dots) and moving each of them around the plane with two degrees of freedom of movement. This is accomplished in *Mathematica* through the use of the Locator command.

Interactivity can occur through the use of sliders as well as cursors simultaneously. For instance, Figure 7 shows the 5th roots of a complex number. The original number can be moved around using the cursor, while the order of the root m can be changed with the slider. The unit circle is shown for reference and to observe what happens to the moduli of the roots when $|z| > 1$ versus when $|z| < 1$.

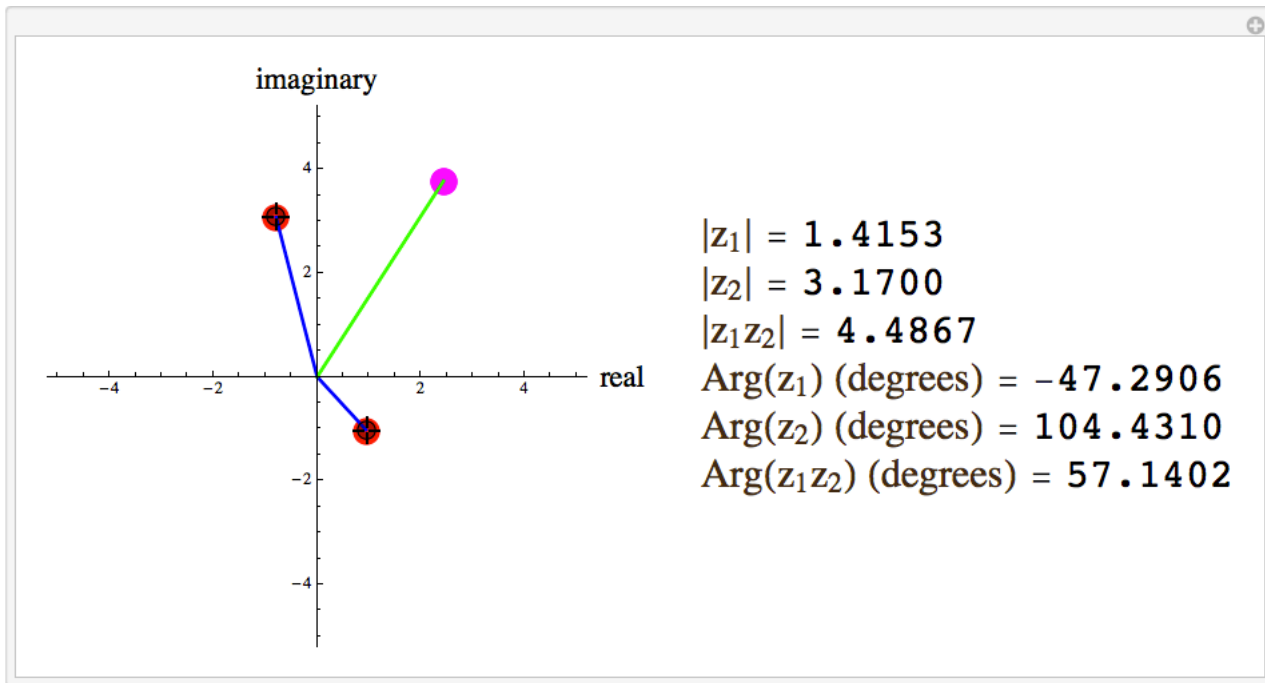


Figure 6: Snapshot of Manipulate animation to illustrate the geometric interpretation of complex multiplication. The interactivity occurs by moving the two cursors around in the plane.

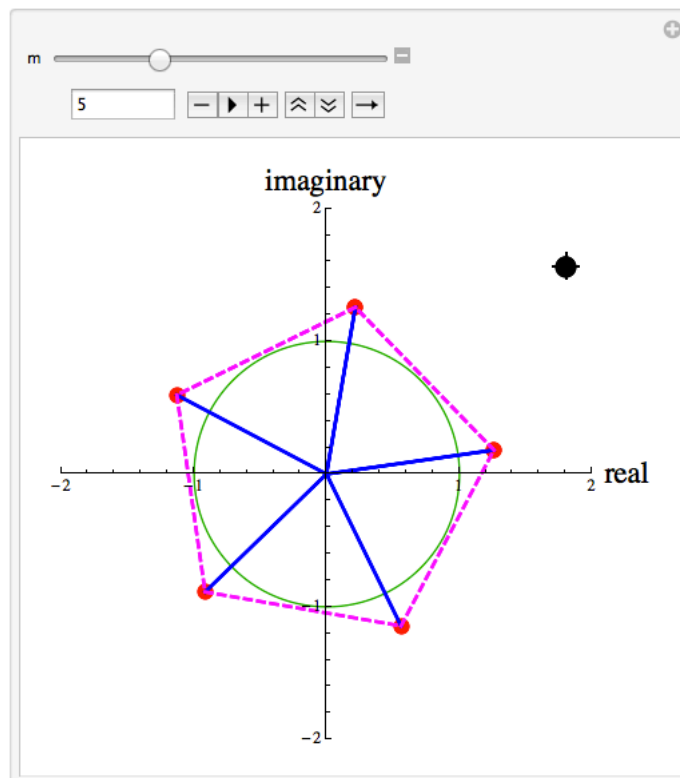


Figure 7: Snapshot of Manipulate animation to illustrate the m^{th} roots of a complex number. The interactivity occurs by moving the one cursor around in the plane as well as by changing the value of m .

From my perspective, the two most important uses of *Mathematica* in basic complex analysis are to illustrate how a complex-valued function of a complex variable can be thought of as a mapping (and how the derivative can give us information about that mapping) and how a complex-valued function of a complex variable has associated vector fields (and how the integral can give us information about those vector fields). For instance, we spent a lot of time exploring the function $w = f(z) = z^2$ as a mapping $f : \mathbb{C} \rightarrow \mathbb{C}$ by using *Mathematica* to visualize the corresponding mapping $(u, v) = \Phi(x, y) = (x^2 - y^2, 2xy) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. The command `NestList` can be used to quickly iterate a function and, combined with the commands `Manipulate`, `Graphics`, and `ListPlot`, create an animation of the orbit of a point under $w = f(z) = z^2$ (see Figure 8).

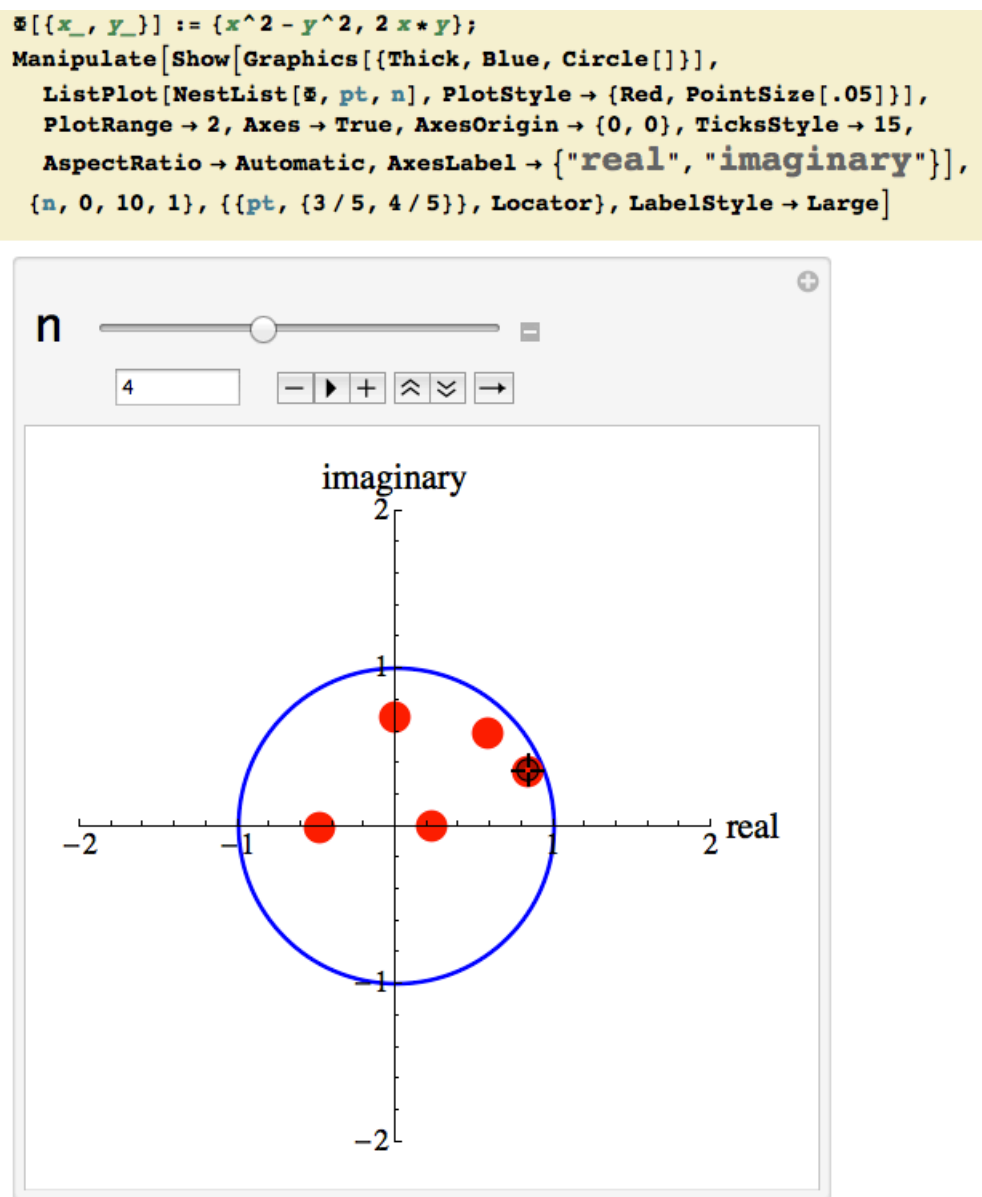


Figure 8: Snapshot of Manipulate animation to illustrate the orbit of a point under $w = f(z) = z^2$.

The Jacobian matrix for the mapping is $J(f) = J(\Phi) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ and the Jacobian determinant is $\text{Jac}(f) = \text{Jac}(\Phi) = u_x v_y - u_y v_x$. When f is analytic, the Cauchy-Riemann equations imply that $\text{Jac}(f) = \text{Jac}(\Phi) = u_x^2 + v_x^2 = |u_x + i v_x|^2 = |f'(z)|^2$ on the domain. For $f(z) = z^2$, we then have $\text{Jac}(f) = \text{Jac}(\Phi) = 4|z|^2$ which means that f shrinks regions completely in the set $|z| < \frac{1}{2}$ and expands regions completely in the set $|z| > \frac{1}{2}$. Figure 9 shows how ParametricPlot and NestList can be combined to visualize images of a small disk inside $\frac{1}{2} < |z| < 1$ under successive application of $f(z) = z^2$. Note how the image initially gets larger but eventually gets smaller as the image points converge to zero.

```

W[[x_, y_]] := {x^2 - y^2, 2 x*y};
Manipulate[
  Show[ParametricPlot[NestList[W, {x, y}, n] /. {x -> r*Cos[t] + c[[1]], y -> r*Sin[t] + c[[2]]},
    {t, 0, 2 π}, PlotStyle -> {{Thick, Red}},
    ParametricPlot[{{.5*Cos[t], .5*Sin[t]}, {Cos[t], Sin[t]}}, {t, 0, 2 π},
    PlotStyle -> {{Thick, Blue}, {Thick, Magenta}}, PlotRange -> 2, AxesOrigin -> {0, 0},
    AxesLabel -> {"real", "imaginary"}], {n, 0, 5, 1}, {{r, .1}, .01, .2},
    {{c, {.6, .6}}, Locator}, LabelStyle -> Large]

```

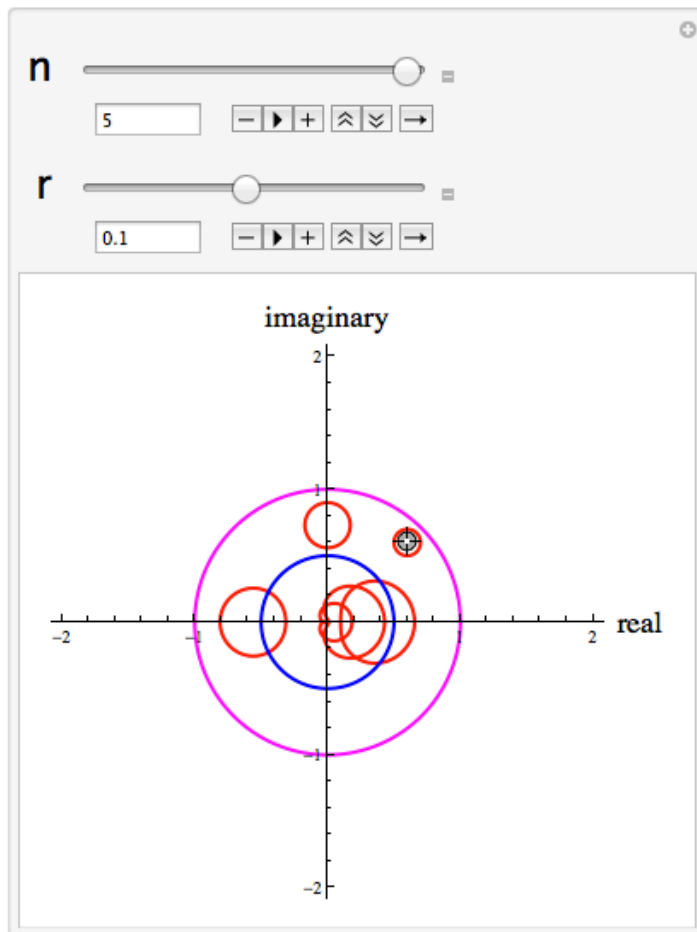


Figure 9: Snapshot of Manipulate animation to illustrate images of a small disk in the set $\frac{1}{2} < |z| < 1$ under iteration of $w = f(z) = z^2$.

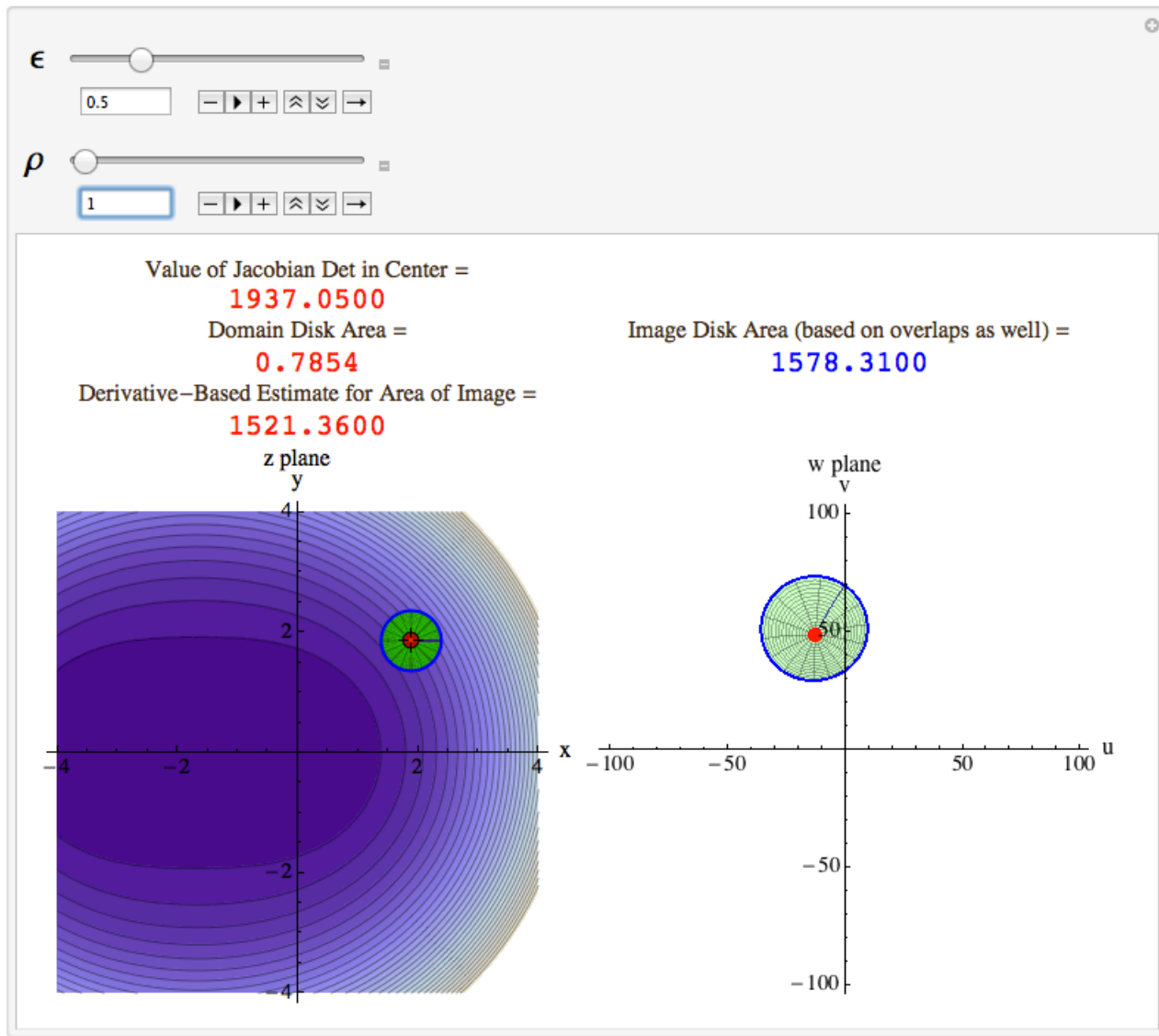


Figure 10: Snapshot of Manipulate animation to calculate the area of the image of a disk under a mapping.

Through the change-of-variables formula for double integrals, we can also program *Mathematica* to help us calculate the exact area of an image of a disk and compare it to the product of $\text{Jac}(f)$ and the area of the domain disk when the disk is small. Figure 10 is the output of such a scheme that I made with my class this spring. The parameter ϵ is the radius of the domain disk and the parameter ρ gives the *relative* radius of the blue circle compared with the whole disk (here we just take $\rho = 1$ in order to focus on the area). The command `NIntegrate` was used to calculate the exact area of the image (including overlaps if the mapping is not one-to-one on the domain). The mapping in this example is $w = f(z) = z^3 + 5z^2$. Note that conformality of the mapping can also be inferred by focusing on how orthogonal families of curves get mapped to orthogonal families of curves. The contour map (level curves) of the function $\text{Jac}(f) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is also shown on the left side of the picture. When moving over this contour map with the cursor (without clicking), the value of $\text{Jac}(f)$ shows up in a box. Darker shading corresponds to lower values of $\text{Jac}(f)$ and lighter shading corresponds to higher values.

The interpretation of the derivative as giving the local “amplitwist” effect of a mapping can also be illustrated (i.e. the interpretation of $dw = f'(z) dz$ as a linear transformation that involves a dilation and a rotation; see the text by Tristan Needham [5]). The output of *Mathematica* code to illustrate this concept is shown in Figure 11. The function is $f(z) = z^2$ so that $f'(z) = 2z$. The short vectors on the left are mapped to vectors on the right and are dilated by a factor of approximately $|f'(2+i)| = 2|2+i| = 2\sqrt{5} \approx 4.5$ and twisted by approximately $\text{Arg}(f'(2+i)) = \text{Arg}(4+2i) = \arctan(\frac{1}{2}) \approx 26.6^\circ$. Once again, we can also see the conformality being illustrated by focusing on the orthogonal sets of curves in each picture.

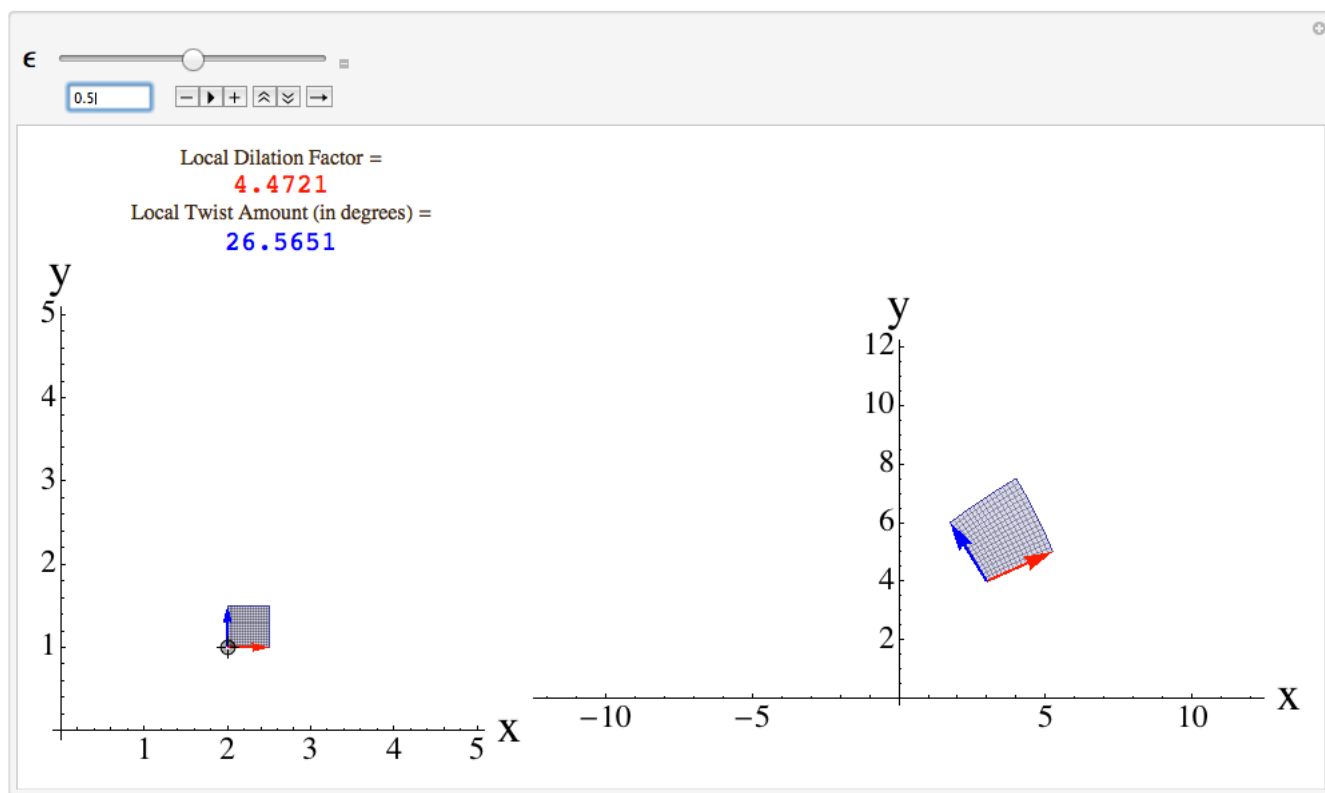


Figure 11: Snapshot of Manipulate animation to illustrate the derivative as representing the local “amplitwist” for $f(z) = z^2$ near $z = 2 + i$.

For complex integration, there are two fruitful perspectives that we discovered as a class that can be illustrated nicely with *Mathematica*. Let $w = f(z) = u(x, y) + iv(x, y)$ be a complex-valued function of a complex variable. If we think of $f(z) dz$ as $(u + iv)(dx + i dy) = (u dx - v dy) + i(v dx + u dy)$, then the integral $\int_{\Gamma} f(z) dz$ can be interpreted as $\int_{\Gamma} \langle u, -v \rangle \cdot ds + i \int_{\Gamma} \langle v, u \rangle \cdot ds$, where $ds = \langle dx, dy \rangle$ and the integrals $\int_{\Gamma} \langle u, -v \rangle \cdot ds$ and $\int_{\Gamma} \langle v, u \rangle \cdot ds$ are ordinary line integrals of the vector fields $\langle u, -v \rangle$ and $\langle v, u \rangle$. As such, they can be interpreted in terms of their circulation relative to the oriented contour Γ [6].

For instance, if f is the non-analytic function defined by $f(z) = f(x + iy) = (x + 2y) + iy^2$ and Γ is the unit circle, oriented counterclockwise, then $\int_{\Gamma} f(z) dz = -2\pi + i\pi$. This means that $\int_{\Gamma} \langle u, -v \rangle \cdot ds = \int_{\Gamma} \langle x + 2y, -y^2 \rangle \cdot ds = -2\pi$ and the average value of $\langle u(z(t)), -v(z(t)) \rangle \cdot z'(t)$ is -1 , while $\int_{\Gamma} \langle v, u \rangle \cdot ds = \int_{\Gamma} \langle y^2, x + 2y \rangle \cdot ds = \pi$ and the average value of $\langle u(z(t)), -v(z(t)) \rangle \cdot z'(t)$ is $\frac{1}{2}$. The output of *Mathematica* code that illustrates this is shown in Figure 12. The background vector fields are very much scaled down, and the vector fields along the unit circle are scaled down as well, but not as much. The value of b gives the current value of t and is set at $b = \pi$ in the snapshot.

Of course, if $w = f(z)$ is analytic in a simply-connected domain, then these circulations will be zero when Γ is a closed contour in the domain because of Green's Theorem and the Cauchy-Riemann equations (both implying Cauchy's Theorem under the extra hypothesis of continuity of the derivatives of u and v). To be more specific, if Γ is a positively-oriented simple closed curve in the simply-connected domain of analyticity and \mathcal{D} is the closure of its interior, then Green's Theorem, under the assumption of continuity of the derivatives of u and v , will produce the equations $\int_{\Gamma} \langle u, -v \rangle \cdot ds = \iint_{\mathcal{D}} (-v_x - u_y) dA$ and $\int_{\Gamma} \langle v, u \rangle \cdot ds = \iint_{\mathcal{D}} (u_x - v_y) dA$, from which the conclusion follows.

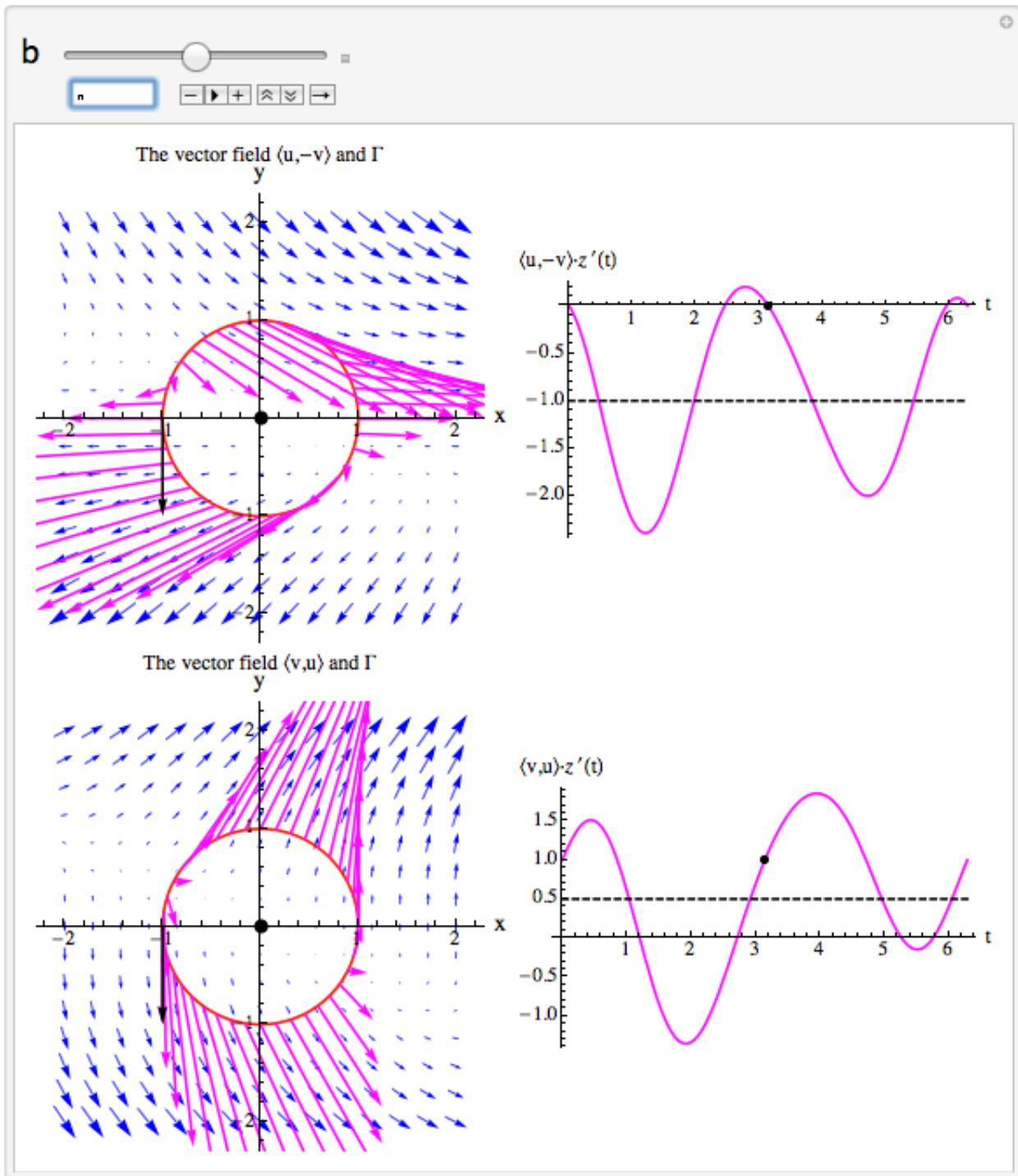


Figure 12: Snapshot of Manipulate animation to illustrate the integral as a combination of the circulation of the vector fields $\langle u, -v \rangle$ and $\langle v, u \rangle$.

Finally, when $w = f(z)$ has an antiderivative $F(z)$ in a domain (which will be true if $f(z)$ is analytic in the domain and if the domain is simply-connected), it is also fruitful to think of $\int_{\Gamma} f(z) dz$ as ΔF , where ΔF is the change in the value of F as z varies over the oriented contour Γ . When f is analytic and Γ is a closed contour in the domain of analyticity, then $\Delta F = 0$ and we confirm Cauchy's Theorem. If $f = u + iv$ and $F = U + iV$, then $\nabla U = \langle U_x, U_y \rangle = \langle U_x, -V_x \rangle = \langle u, -v \rangle$ and $\nabla V = \langle V_x, V_y \rangle = \langle V_x, U_x \rangle = \langle v, u \rangle$ and we can visualize $\Delta F = \Delta U + i\Delta V$ by focusing on the three-dimensional graphs of $U(x, y)$ and $V(x, y)$.

For example, if $f(z) = 3z^2 = (3x^2 - 3y^2) + i(6xy)$, then $F(z) = z^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$. If Γ is the oriented line segment from $z = 0$ to $z = 1 + 2i$, then $\Delta U = 1^3 - 3 \cdot 1 \cdot 2^2 = 1 - 12 = -11$ and $\Delta V = 3 \cdot 1^2 \cdot 2 - 2^3 = 6 - 8 = -2$ (and therefore $\int_{\Gamma} f(z) dz = -11 - 2i$). In Figure 13, we see by looking at the red curve that these values do seem to be correct.

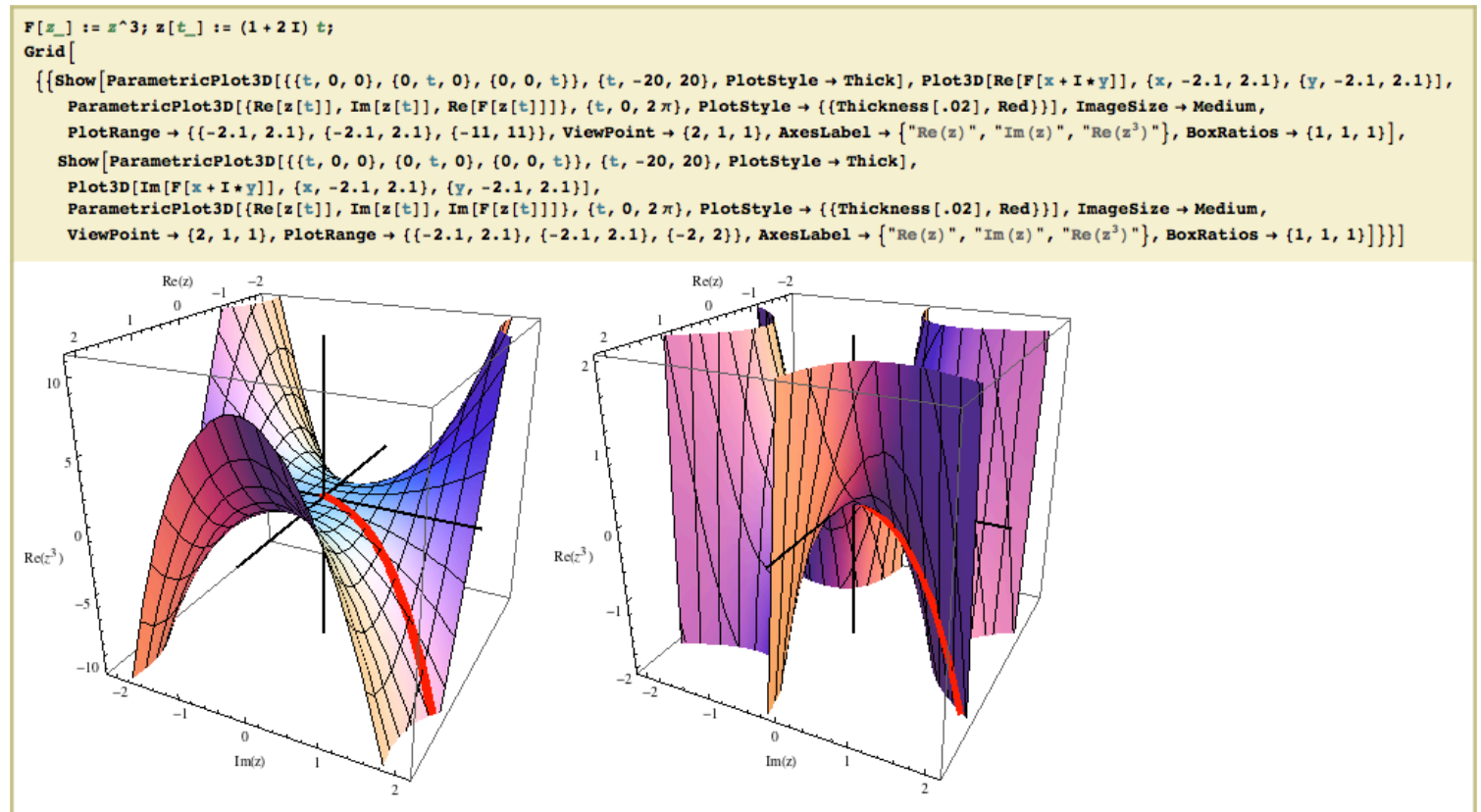


Figure 13: Illustrating the value of the Integral $\int_{\Gamma} 3z^2 dz$ in terms of ΔU and ΔV , where $\Delta F = \Delta U + i\Delta V$ and $F(z) = z^3$.

This mode of thinking can also help us make sense of integration of meromorphic functions around contours containing poles giving integrals that are nonzero and dependent on the residues of the function at the poles. As the most fundamental example of such a situation, if $f(z) = \frac{1}{z}$, then we know that f is analytic on $\mathbb{C} - \{0\}$ but that, for example, the principal value of the logarithm $F(z) = \text{Log}(z) = U + iV = \ln|z| + i\text{Arg}(z)$ is only an antiderivative of f on the slit domain $\mathbb{C} - \{x + iy : x \leq 0, y = 0\}$. If Γ is, for instance, the unit circle, oriented counterclockwise, then we also know that $\int_{\Gamma} f(z) dz = 2\pi i$. This cannot be strictly interpreted in terms as $\Delta F = \Delta U + i\Delta V$. It can, however, *almost* be interpreted in

this way. If we let w^+ be the limiting value of $F(z)$ as z approaches -1 from above the real axis in the complex plane and let w^- be the limiting value of $F(z)$ as z approaches -1 from below the real axis in the complex plane, then $w^+ = \pi i$ and $w^- = -\pi i$ and $w^+ - w^- = 2\pi i$. Furthermore, if we let v^+ be the limiting value of $V(z) = \text{Arg}(z)$ as z approaches -1 from above the real axis in the complex plane and let v^- be the limiting value of $V(z) = \text{Arg}(z)$ as z approaches -1 from below the real axis in the complex plane, then $v^+ = \pi$ and $v^- = -\pi$ and $v^+ - v^- = 2\pi$ and $w^+ - w^- = (v^+ - v^-)i$. In other words, if we think in terms of limiting values, we can still think about $\int_{\Gamma} f(z) dz$ as being “essentially” given by $\Delta F = \Delta U + i\Delta V$ and we can visualize it with the 3-dimensional plot in Figure 14.

```

F[z_] := Log[z]; z[t_] := E^(I*t);
Grid[
  {{Show[ParametricPlot3D[{{t, 0, 0}, {0, t, 0}, {0, 0, t}}, {t, -5, 5}, PlotStyle -> Thick],
    Plot3D[Re[F[x + I*y]], {x, -1.2, 1.2}, {y, -1.2, 1.2}],
    ParametricPlot3D[{Re[z[t]], Im[z[t]], Re[F[z[t]]}], {t, 0, 2 pi}, PlotStyle -> {{Thickness[.02], Red}},
    ImageSize -> Medium, PlotRange -> {{-1.2, 1.2}, {-1.2, 1.2}, {-2, 1}}, ViewPoint -> {2, 1, 1},
    AxesLabel -> {"Re(z)", "Im(z)", "Re(Log(z))"}, BoxRatios -> {1, 1, 1}],
  Show[ParametricPlot3D[{{t, 0, 0}, {0, t, 0}, {0, 0, t}}, {t, -5, 5}, PlotStyle -> Thick],
    Plot3D[Im[F[x + I*y]], {x, -1.2, 1.2}, {y, -1.2, 1.2}],
    ParametricPlot3D[{Re[z[t]], Im[z[t]], Im[F[z[t]]}], {t, 0, 2 pi}, PlotStyle -> {{Thickness[.02], Red}},
    ImageSize -> Medium, ViewPoint -> {2, 1, 1}, PlotRange -> {{-1.2, 1.2}, {-1.2, 1.2}, {-pi, pi}},
    AxesLabel -> {"Re(z)", "Im(z)", "Im(Log(z))"}, BoxRatios -> {1, 1, 1}}]}]}

```

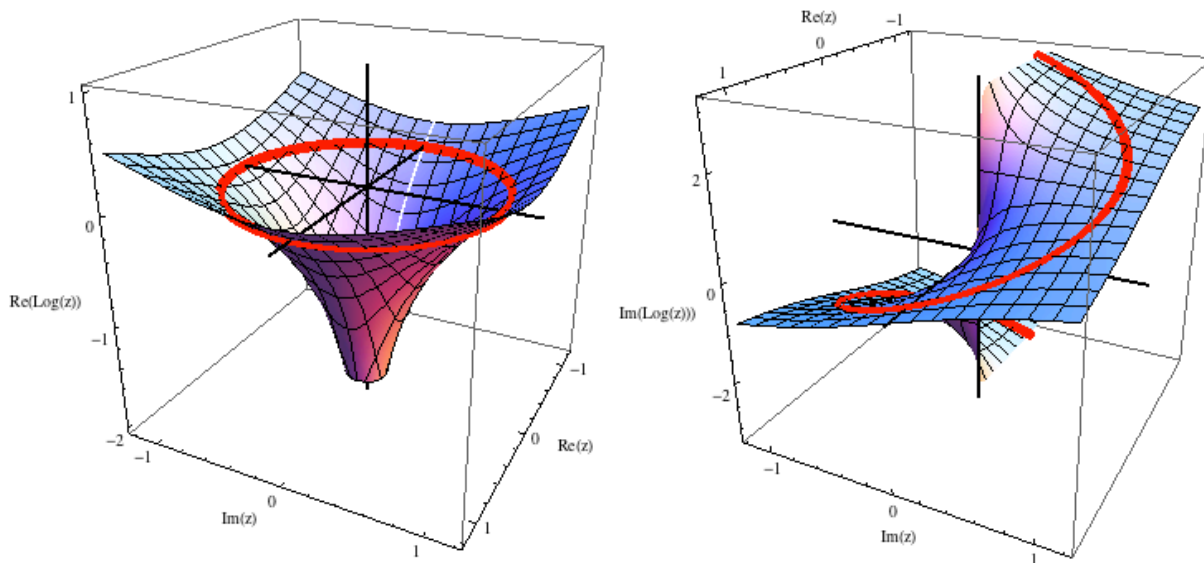


Figure 14: Illustrating the value of the Integral $\int_{\Gamma} \frac{1}{z} dz$ in terms limiting values of ΔU and ΔV , where $F(z) = U(z) + iV(z) = \text{Log}(z) = \ln|z| + i\text{Arg}(z)$.

This is even more significant in more challenging examples, such as for the function $f(z) = \frac{1}{(z-i)(z-2)}$. The residue of this function at $z = i$ is $-\frac{2}{5} - \frac{i}{5}$ and the residue at $z = 2$ is $\frac{2}{5} + \frac{i}{5}$. If Γ_1 is $|z - i| = 1$ (positively oriented), Γ_2 is $|z - 2| = 1$ (positively oriented), and Γ_3 is $|z| = 3$ (positively oriented), then the Residue Theorem gives $\int_{\Gamma_1} f(z) dz = \frac{2\pi}{5} - \frac{4\pi}{5}i \approx 1.25664 - 2.51327i$, $\int_{\Gamma_2} f(z) dz = -\frac{2\pi}{5} + \frac{4\pi}{5}i \approx -1.25664 + 2.51327i$, and $\int_{\Gamma_3} f(z) dz = 0$. If we let $U(z) + iV(z) = F(z) = \int f(z) dz = \left(\frac{1}{5} - \frac{2i}{5}\right) \tan^{-1}\left(\frac{z-2}{1+2z}\right) + \left(\frac{1}{5} + \frac{i}{10}\right) \text{Log}((z-2)^2) - \left(\frac{1}{5} + \frac{i}{10}\right) \text{Log}(1+z^2)$ (determined with *Mathematica's* symbolic computational abilities; *Mathematica* can calculate residues and Laurent series expansions as well) over an appropriate domain with appropriate branch cuts, then Figure 15 shows that each integral can be thought of as “essentially” $\Delta U + i\Delta V$, where these quantities ignore the discontinuities and also are calculated using appropriate limiting values. It is difficult to “see” the correct changes in this example, but it can be confirmed more convincingly by using *Mathematica's* ability to allow the user to rotate these pictures using the cursor.

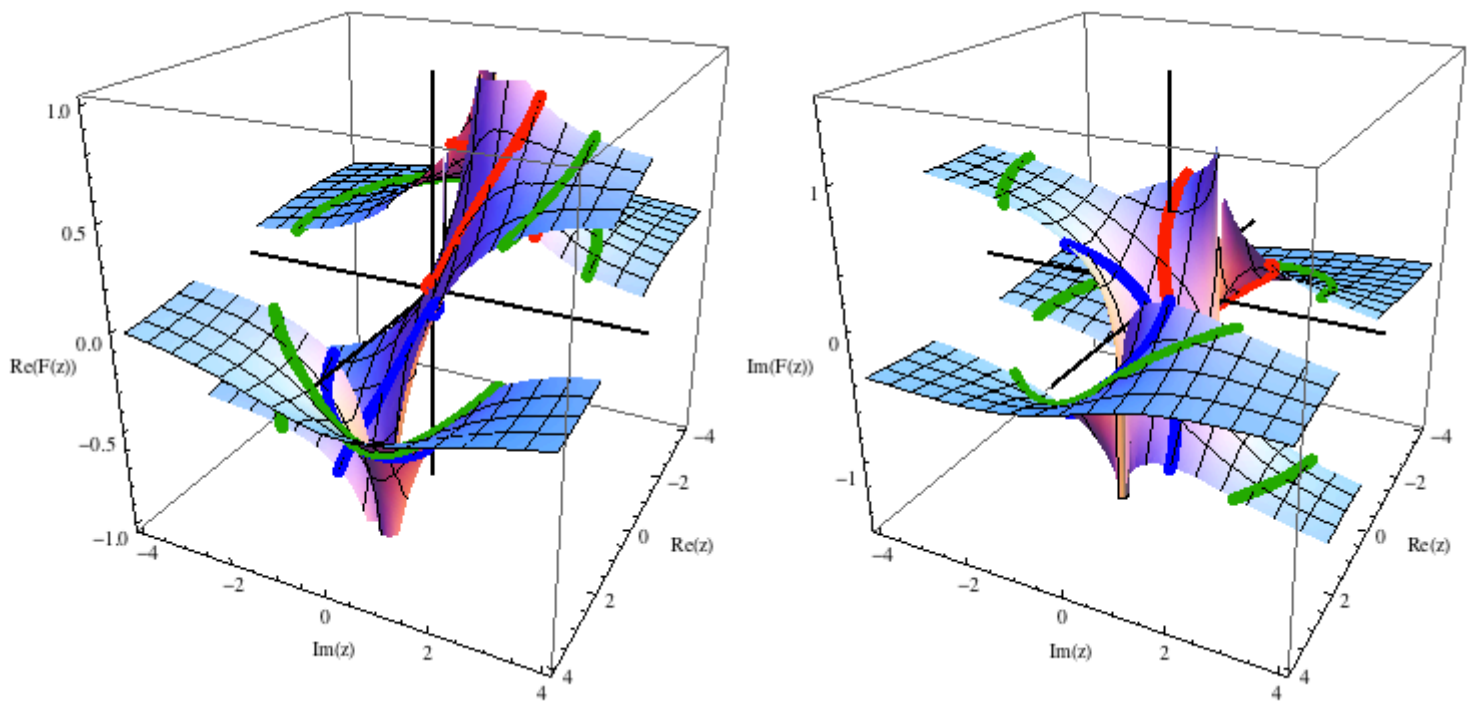


Figure 15: Illustrating the value of the integrals $\int_{\Gamma_j} \frac{1}{(z-i)(z-2)} dz$ for $j = 1, 2, 3$ in terms of using appropriate limiting values to calculate ΔU and ΔV , where $F(z) = \int f(z) dz$ over an appropriate domain.

Other uses of *Mathematica* in my complex analysis course included: (1) visualizing how the plane can be mapped onto the Riemann sphere, (2) visualizing the definition of an open set (by considering simple examples where it is easy to calculate how the radius of a neighborhood of each point can be appropriately shrunk as the point approaches the boundary to keep the neighborhood inside the set), (3) illustrating the truth of the Cauchy-Riemann equations for analytic functions by focusing on how the local contour maps of u and v are approximately rotations of each other (up to an additive constant) near each point, (4) illustrating the truth the maximum principle for harmonic functions, (5) illustrating the non-conformality and non-injectiveness of a complex analytic function near a critical point, and (6) illustrating the Gauss-Lucas Theorem about how the critical points of a polynomial lie in the convex hull of the zeros of the original polynomial (it is also interesting to graph the level sets $u = 0$ and $v = 0$ simultaneously).

The Wolfram Demonstrations Project [9] (<http://demonstrations.wolfram.com/>) also has many resources for constructing models and illustrating concepts in many subjects, including complex analysis. As of early June in 2013, a search of “complex analysis” at that website produced 173 such “demonstrations” that can be downloaded and used in class (with appropriate credit given). I am particularly fond of code to construct the Mandelbrot set and associated Julia sets.

4 Use of Video Lectures

As stated in the introduction, my goal in all this is to make my classrooms “flipped” and to have the students ultimately be constructing these kinds of pictures and models for themselves. However, the vast majority of students still need basic instruction, as well as more help with the *Mathematica* code than I can give them during our class periods. Therefore, an important part of what I am working on to flip my classrooms is to make video lectures, each of which I typically try to make about ten minutes in length. The lectures focus on: (1) basic computational methods and concepts for each course as well as (2) constructing *Mathematica* code to illustrate these methods and concepts and to enable better use of *Mathematica* in the classroom.

Because of the difficulty I had with my computer this spring, I was only able to fully implement this vision of flipping the classroom for the first couple weeks, when we were in Chapter 1 of our textbook [7]. As of early June in 2013, I uploaded twenty-eight such lectures for Complex Analysis onto my channel on *YouTube* [10] (I plan to make more videos related to this subject the rest of the summer of 2013, as well as more videos about other subjects). The URL for my channel is <http://www.youtube.com/user/billkinneymath> and you can also find my channel by doing a Google search “Bill Kinney Math”. If you watch these videos you will get a better sense of what I am trying to do than you’ll get from my attempt to explain it here.

Would I ever want to use someone else’s lectures? Yes, if I like what someone else is doing, I would make use their lectures (though I have not taken much time to look beyond some of the Khan Academy lectures). However, though it is very time-consuming, I very much enjoying making lectures in this way. Because of this, I think, for the most part, I will rely on my own lectures.

5 Conclusion

I hope that I have inspired you to consider flipping your classroom and to consider using *Mathematica* in your teaching, whether you flip your classroom or not. Please let me know if you have any questions or if you have ideas you would like to share with me. I am especially interested in calculus-related projects that can be modeled and analyzed with *Mathematica*.

6 References

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