The Deconstruction of Mathematics

A criticism of Reuben Hersh's What Is Mathematics, Really? *and the Humanist Philosophy of Mathematics*

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> "My readers know how impatient I am with some pragmatists, phenomenologists, and subjective idealists of various schools who heap scorn on the notion that mathematical structures are 'out there' with a reality that is not mind dependent. For these thinkers, mathematical reality is located within human experience. Like...the overwhelming majority of eminent mathematicians past and present, I am a Platonist in the sense that I believe mathematical patterns are discovered, not invented. Of course, they are still invented, in a sense. Everything humans do and say is what humans do and say. Mathematics obviously is part of human culture, but to say so is to say something utterly trivial."
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> —Martin Gardner (1996, p. 92)

I

Mathematics, as an academic discipline, has stood for many years as the last bastion against a growing tide of intellectual relativism that has become all but ubiquitous. More recently, however, efforts have been made to "humanize" mathematics by advocating a social-constructivist approach to the philosophy of mathematics, both in practice and education. This attempt to bring mathematics in line with the wider body of post-modernistic thinking may have unfortunate far-reaching consequences, both educationally and in furthering the societal cancer of moral relativism. Among the leaders of this movement are Paul Ernest (*Social Constructivism in the Philosophy of Mathematics*, 1997 and *The Philosophy of Mathematics Education*, 1991) and Reuben Hersh (*What Is Mathematics, Really?*, 1997). This paper is intended to serve as a critical response to Hersh's book, and in the process a defense of Platonism.

What Is Mathematics, Really? opens with an astonishing dialog between Hersh and a 12-year old girl about the kinds of questions that can begin to help us understand what the philosophy of mathematics is all about, and what is at stake in the process. I have already participated in several dramatic readings of this passage in classroom situations, with the purpose of setting the stage for further discussion. It has proven to be an effective tool in motivating students.

The remainder of the book is organized into two sections followed by a summary chapter. The first section, in five chapters, lays out the core of what a "humanistic" philosophy of mathematics would be like, and the motivations for preferring it to any and all of its historical predecessors. Unfortunately, Hersh only manages to unmask himself as either disingenuous and agenda-driven (the most likely scenario), or completely incompetent, lacking an understanding of the most fundamental issues and distinctions in science and philosophy (hard to imagine of someone with his credentials). In contrast, the second section is a remarkable survey of the history of mathematical philosophies, complete with summaries of the key players and their roles and contributions. These seven chapters are a tour-de-force that will certainly become regarded as a classical resource in the philosophy of mathematics. In the summary chapter, after displaying a self-graded report card, Hersh attempts to connect the importance of a humanistic approach to educational objectives and political ideals.

II

As the 20th century draws to a close, rationalists and materialists are experiencing an increasing crisis of faith on a multitude of fronts. The widespread interest in Eastern philosophies and meditation techniques amongst scientists (particularly physicists) and philosophers; the incredible lifting of the taboo against consciousness research just in the last decade; the publication by the popular press of John Horgan's best-selling book *The End of Science: Facing the Limits of Knowledge in the Twilight of the Scientific Age* (Horgan, 1996); the recognition that all three of the major "foundational" philosophies of mathematics have fallen short of their promises—these are but some of the symptoms of the general failure of rationalism and materialism to provide satisfactory answers to the central philosophical questions. Humanism, within its larger context of post-modern pluralism, remains, then, the final hope for a naturalistic account of knowledge, language, consciousness, intellect, and the creative spirit of the human species.

Yet, to borrow Hersh's attribution of Platonism (Hersh, 1997, p. 11), most of this pluralism is half-hearted, shamefaced. For once a position of relativism or fallibilism is adopted there is an unavoidable, inherent contradiction. How, for example, is it possible to take a straight-faced position that the only absolute is the non-existence of absolutes? Thus is Pandora's box opened, admitting all monstrous ideas, to the annihilation of knowledge. A much more palatable position is that most philosophies have an element of truth, yet truth itself exists, is absolute, is infallible, and that the hard work is to sift through and discern these boundaries.

Considering all of this, the premise that I work from is that ***all knowledge is contingent upon faith.*** Establishing any adequate foundation for this premise would require a work of much larger scope than this paper. Very little work has been done to explore the philosophical role and applicability of faith outside the scope of Judeo-Christian theology. While my premise is certainly consistent with St. Anselm's notion that understanding is impossible without faith, Karl Barth's identity of faith with knowledge[1], and Abraham Heschel's concept of faith emerging out of *knowledge by inacquaintance*[2], I intend it as a much more general thesis, encompassing not only theology, but all of philosophy.

Allow me, however, to motivate it briefly, particularly with regard to mathematics. As Hersh acknowledges throughout his book, one's philosophy, politics, theology (or lack thereof), educational pedagogy, etc., are inextricably interwoven into a personal paradigmatic view of reality. Hersh painstakingly details how, up until the 19th century, the heart of mathematics was strongly tied to religion. From the mysticism of the Pythagoreans through the metaphysics of Kant, the divine nature (or origin) of numbers was recognized as a given. Theism reigned. During the 1800's the crisis of faith embodied in the enlightenment began to corrode the theistic foundations of mathematics. Mathematicians began to see philosophy's primary role as that of providing a systematic and secure foundation

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[1] See Barth, 1959.

[2] "The ultimate insight is the outcome of ***moments*** when we are stirred beyond words, of instants, of wonder, awe, praise, fear, trembling and radical amazement; of awareness of grandeur, of perceptions we can grasp but are unable to convey, of discoveries of the unknown, of moments in which we abandon the pretense of being acquainted with the world, of *knowledge by inacquaintance.* It is at the climax of such moments that we attain the certainty that life has meaning, that time is more than evanescence, that beyond all being there is someone who cares." (Heschel, 1955)

for mathematical certainty. Thus 20th century philosophy of mathematics has been almost exclusively one of 'foundationism,' a response to the secularization of mathematics. Faith in the divine was replaced, after Descartes, with a faith in reason, which eventually decayed into a faith in deduction and logic.

None of foundationism's branches, logicism, formalism, or constructivism (which includes intuitionism) has succeeded in filling the void left when religion and mathematics diverged. Foundationism has faltered because it assumed that deduction and logic 'go all the way down.' In other words that the foundations of mathematics lie internally within mathematics (or logic) itself. Lacking any other modernistic philosophies, the question becomes, 'Wherein are the proper foundations of mathematics?' The only clear options are either a return to the theistic foundations of early mathematics, or the construction of a post-modern philosophy of mathematics. Unfortunately, in choosing the latter, Hersh has found it necessary to subvert an equally fundamental canon of mathematics: Platonism.

In the five chapters that Hersh devotes to the construction of a humanistic philosophy of mathematics (henceforth humanism), he fails miserably to perform any adequate analysis or synthesis of ideas. He repeatedly exhibits category errors, confusion of conceptual layers, ignorance of large bodies of literature, and an inability to make fundamental distinctions. His assertions are the result of begging the question, logical fallacies, straw men, false analogies, linguistic smoke and mirrors, garden paths, and wishful thinking. In the next section I will make a point-by-point critique of Hersh's defense of humanism, providing examples of the aforementioned errors.

III

In chapter 1 Hersh believes he can make room for humanism by exposing the weaknesses of Platonism. Yet all he actually accomplishes is to discard it out of hand as inconsistent with materialism:

> "Yet most of this Platonism is half-hearted, shamefaced. We don't ask, How does this immaterial realm relate to material reality? How does it make contact with flesh and blood mathematicians? We refuse to face this embarrassment." (p. 11)

His solution, however, is not any less embarrassing. He acknowledges that mathematics is neither physical nor mental in nature, agreeing on this point with Platonists. Rather than a Platonic realm, however, he posits that "A world of ideas exists, created by human beings, existing in their shared consciousness." (p. 19) He calls this world the social-cultural-historical reality.

His arguments rely on subtle linguistic tricks and an intentional confusing of key distinctions in mathematics and science. Consider the following example. Martin Gardner has often defended Platonism by observing, "if two dinosaurs met two others in a forest clearing, there would have been four dinosaurs there—even though the beasts were too stupid to count and there were no humans around to watch." (Gardner, 1996, pp. 92-3) Hersh responds to this by claiming that Gardner is confusing the adjective "two" with the noun "two":

> "In 'two dinosaurs,' 'two' is a *collective adjective*...In contrast, 'Two is prime...' is a statement about the pure numbers of elementary arithmetic...The collective adjectives or 'counting numbers' are finite. There's a limit to how high anyone will ever count. Yet there isn't any last counting number. ...In pure arithmetic, these two properties—finiteness, and not having a last—are contradictory. This shows that counting numbers aren't the pure numbers.

> Consider the pure number $10^{(10^{10})}$. We easily ascertain some of its properties, such as: 'The only prime factors of $10^{(10^{10})}$ are 2 and 5.' But we can't count that high. In that sense, there's no counting number equal to $10^{(10^{10})}$." (p. 15)

But this is not only linguistically invalid, it ignores the difference between the theoretical and the practical. Linguistically, the following parallel exposes the logical flaws in Hersh's argument. In 'blue sky,' 'blue' is a *color adjective*. In contrast, 'Blue lies between green and violet' is a statement about the colors of the color spectrum. But it doesn't matter. Even in the time of the dinosaurs, the earth's atmosphere scattered light with short wavelengths more than light with long wavelengths, i.e., there was a blue sky. Whether 'two' or 'blue' is used as an adjective or a noun is irrelevant to its semantic referent. In either case it is a pure number or the visual experience of a particular frequency of light. Furthermore, I am not convinced that there is any difference between counting numbers and pure numbers. Theoretically, there are an infinite number of natural (pure) numbers. Theoretically, there are also an infinite number of collective adjectives. In practice, there are a finite number of natural numbers; there is a limit to the finite magnitude that will ever be conceived. In practice there is a limit to how high anyone will ever count. There is nothing, in principle, to prevent me from counting to $10^{(10^{10})}$. Only my mortality presents an obstacle. Hersh pretends that intractability is an issue of theoretical rather than practical concern. He makes the same mistake on the very next page:

> "From living experience we know two facts:
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> *Fact 1: Mathematical objects are created by humans. Not arbitrarily, but from activity with existing mathematical objects, and from the needs of science and daily life.*
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> *Fact 2: Once created, mathematical objects can have properties that are difficult for us to discover.* This is just saying there are mathematical problems which are difficult to solve. Example: Define x as the 200th digit in the decimal expansion of $23^{(45^{6789})}$. x is thereby determined. Yet I have no effective way to find it." (Hersh, p. 16)

Consider the following parallel: Define y as the number of planets orbiting the star in the Milky Way galaxy that is furthest from our sun (given the dynamic nature of the galaxy we can fix the date to remove any ambiguity, say on January 1, 2001 AD). y is thereby determined. Yet I have no effective way to find it. This is not because the galaxy is part of some social-cultural-historical reality. It is simply because I don't have the mental, physical, or technological faculties to measure y. My difficulty is entirely practical and has no theoretical or philosophical implications.[3]

One further word of caution regarding the example above: when Hersh asserts *as a fact* that mathematical objects are created rather than discovered, he is begging the question, predisposing the reader to accept the humanist philosophy over Platonism.

The primary challenge Hersh faced in chapter 1 was to expose the axiomatic, unprovable, articles of faith of Platonism while disguising the articles of faith of humanism as rational. To a large degree his failure is in not recognizing that the distinction between the epistemology and ontology of mathematics is natural rather than paradoxical. I will simply let his own words communicate the degree to which he failed:

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[3] In fact, his example is even more misleading because of his correct use of the technical word "effective." What this means, is that there is an entirely trivial way to find x. The algorithm could be coded in just a few lines of, say, C++. An effective procedure in computer science is one that is tractable (computable in a *reasonable* amount of time, where reasonable is usually taken as polynomial). Since the trivial algorithm is intractable, Hersh is technically correct, but not very clear in communicating the source of the difficulty.

> "Why do mathematicians believe something so unscientific, so far-fetched as an independent immaterial timeless world of mathematical truth?" (p. 11)
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> "Ideal entities independent of human consciousness violate the empiricism of modern science." (p. 12)
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> "Mathematics is part of human culture and history, which are rooted in our biological nature and our physical and biological surroundings. Our mathematical ideas in general match our world for the same reason that our lungs match earth's atmosphere." (p. 17)
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> "I say...any mathematical object you like...exists at the social-cultural-historic level, in the shared consciousness of people (including retrievable stored consciousness in writing). In an oversimplified formulation, 'mathematical objects are a kind of shared thought or idea.' " (p. 19)
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> "Between a 4-cube and the idea of such, there is confusion. Why? Because we have nowhere to point, to show a 'real' 4-cube as distinct from the idea of a 4-cube." (p. 20)

So on the one hand Hersh scoffs at an independent immaterial **ontological** realm as unscientific and far-fetched, while simultaneously proffering an independent immaterial **epistemological** realm as less mysterious. It is ironic that he would prefer that mathematics reside in the consciousness of people than elsewhere, when consciousness remains impervious to empirical, rationalist explanation. Likewise, science has had no success in developing a natural theory of shared thought (the transmission of semantic content from one consciousness to another). Finally, for Platonists there is no confusion between a 4-cube and the idea of a 4-cube. The former is ontologically real, while the other is an epistemological approximation or representation.

Hersh wraps up the first chapter with a puzzling section in which he first attempts to downplay the importance of philosophy, and then characterizes humanism as unique amongst philosophies of mathematics in allowing mathematics to be fallible. In this section he initiates a pattern of confusing the nature of and relationships between the various philosophies. Rather than consistently identifying logicism, formalism, and contructivism (of which intuitionism is a kind) as branches of foundationism, and Platonism as much larger in scope than these (and not necessarily hostile to any of them), he selectively pits them against each other in different combinations and permutations throughout the book.

The second chapter is devoted to establishing a laundry list of thirteen criteria by which various philosophies of mathematics may be judged. Rather than addressing the specifics of these criteria here, I will wait until my discussion of Hersh's final chapter in which he uses them to grade his own philosophy. However, there is one peripheral issue he raises that is worthy of some attention.

Possibly, the source of Hersh's weak and degenerative view of mathematics may be summed up by his statement, "My first assumption about mathematics is: It's something people do. An account of mathematics is unacceptable unless it's compatible with what people do, especially what mathematicians do." (p. 30) The problem is that he is conflating result with process. The objects of mathematics are the results, the subject matter, of the human process of mathematics. Just as the cosmos, as the subject matter of astronomy and cosmology, is distinct from the scientific method, that which defines the process of these disciplines. The philosophy of mathematics, historically and rightly, has been only concerned with the objects of mathematics. The other ***is*** a social-cultural-historic phenomenon, but is irrelevant to the ***nature*** of mathematical constructs. This is the heart of the disagreement, ultimately a difference of faiths that cannot be resolved by an appeal to reason. Hersh's humanism claims that the only 'reality' is the process of doing mathematics.

C. P. Snow (1969) talked of two cultures: the humanities and the sciences, existing side by side in the academy but having no common ground for discourse. Humanists, desiring to subsume all of academia under their umbrella, have proposed various means of bridging this chasm. For example, E. O. Wilson has suggested in his latest book, *Consilience* (1998), a reductionist approach: explaining all of the 'humanities' in terms of the deeper truths of genetics and natural selection. Hersh is taking the opposite reductionist approach: reducing mathematics to a social-cultural-historic reality—empire-building on behalf of anthropology and sociology. In both cases, the solution is to subsume one under the umbrella of the other, an endeavor that only serves to fan the flames of contention and widen the gap between the two cultures.

I would prefer a reconciliation that doesn't require the obliteration of either the humanities or the sciences. In his critical response to *Consilience*, renowned humanist Stephen Jay Gould concludes with the following statement:

> "The humanities cannot be conquered, engulfed, subsumed or reduced by any logic of argument, or by any conceivable growth of scientific power. The humanities, as the most glorious emergent properties of human consciousness, stand distinct and unassailable. Any complete human life, any hope for attaining the Old Testament ideal of wisdom, must join the factuality of scientific understanding to the moral and aesthetic inquiry of our most particularly human capacities. Why not try for perpetual balance and communion between these disparate sources of wisdom: 'Whither thou goest, I will go'?" (Gould, 1998)

While I agree with the spirit of Gould's plea, I fundamentally differ with him in his dogmatic commitment to materialism: "Furthermore, though I recognize the impossibility of scientifically testing such a proposition, I strongly suspect that all the glorious (and unseemly) capacities of the human brain arise from material properties of evolved neurology, and not by any infusion or 'suraddition' of an ineffable property from some independent domain that might be called 'spiritual.'" (Gould, 1998) To the contrary, I see spirituality and mysticism, in its most general sense rather than as a narrower category like 'eastern mysticism,' as the best hope for bridging the gap between the two cultures and establishing the balance and communion Gould suggests. Gould's (and Hersh's) humanism, which denies the need for faith in anything higher than humanity itself, is impoverished and bankrupt. This is my dogma.

Myths/Mistakes/Misunderstandings is Hersh's title for his third chapter, wherein he spins his own myths, makes mistakes, and exposes his own misunderstandings. His most consistent mistake in this chapter is a practice he condemns later in chapter 10: "As some philosophers have observed, we owe many errors to the abuse of words." (p. 193)

He begins the chapter with a useful metaphor of institutional 'front' and 'back'. Restaurants have a front, which is public, and in which the customer is served (the result), and a back, which is private, and in which all is made ready and the food is prepared (the process). He extends this metaphor to universities, where "classrooms and libraries are the front, where the public (students) is served. The chairman's and dean's offices are the back, where the products (classes and courses) are prepared."[4] (p. 35) For Hersh, however, this is a disingenuous metaphor. As we have discussed already, by conflating result and process Hersh doesn't believe the food is real. It is merely a convenient social construct. To disabuse his application of this metaphor he would have to admit that a philosophy of the social process of mathematics is largely independent of a philosophy of the nature of mathematical objects.

The worst abuse of language Hersh commits is in his use of the word myth. His explication of myth positively positions him on the science side of the 'two cultures' gap, exhibiting his ignorance of the literary and theological

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[4] I would be less surprised at his distorted view of mathematics if this were true of his own university. I have tended to associate myself more with institutions where the faculty determined the curriculum. ; ^)

nature and role of myth. There is for example, nothing mythopoeic about the concepts he alleges as mythical in the traditional view: unity, universality, certainty, and objectivity. On the other hand he completely misses the mythopoeic quality of the beliefs of the Pythagoreans. Rather, he identifies a myth as an intentionally constructed falsehood, relating it to the front/back metaphor:

> "A myth isn't bad *per se*. It has allegorical or metaphorical power. It may increase the customer's enjoyment. It may be essential for the performance.
>
> "The myth of the divine right of kings was useful. So are the myths of Christmas, Easter, and those of other religions.
>
> "There's an unwritten criterion separating the professional from the amateur, the insider from the outsider: The outsider is taken in by the myths. The insider is not." (p. 37)

In these three brief paragraphs Hersh manages to juxtapose several disparate senses of the word. In one fell swoop he casts all professional theologians, clergy, and ministers as disingenuous disbelievers of their own doctrine.

More to the point, the four concepts Hersh identifies as general myths are all *ontological* claims:

1. The **unity** of mathematics: it is a single inseparable whole.
2. The **universality** of mathematics: it is always the same.
3. The **certainty** of mathematical truth: it is absolute.
4. The **objectivity** of mathematics: it is independent of context or epistemology.

All of his insider arguments against these are *epistemological*. He dismisses unity due to lack of communication or understanding between different branches of mathematics. He claims universality is dependent on our ability to communicate with alien species. Certainty is supposedly undermined by a lack of rigor practiced by many mathematicians. He doesn't even manage to find an objection to objectivity. Instead he plays a linguistic sleight of hand and redefines objectivity as having to do with taste rather than truth.

The argument becomes even more twisted with the following abuse of analogy:

> "Mathematicians want to believe in unity, universality, certainty, and objectivity, as Americans want to believe in the Constitution and free enterprise, or other nations in their Gracious Queen or their Glorious Revolution. But while they believe, they know better.
>
> "To become a professional, you must move from front to back. You get a more sophisticated attitude to myth." (p. 39)

What a dark and sinister worldview. In Hersh's view mathematical ontology is never more than, in fact is identical to, mathematical epistemology. So the 'reality' of mathematics ebbs and flows with the sum of human knowledge. This, in turn, justifies the creation of convenient fictions for public consumption. He has become the old man in Oz, hiding behind a curtain of shame.

Hersh next turns his attention to the dilemma that many modern mathematicians have faced: how to retain the belief in an independent Platonic realm, yet deny any philosophical problems with this by also adopting a formalist stance. This dilemma, however, only plagues materialist Platonists. Hersh accurately identifies the solution to the dilemma:

> "Platonism in the strong sense—belief in the existence of ideal entities, independent of or prior to human consciousness—was tenable with belief in a Divine Mind. Once mysticism is left behind, once scientific skepticism is focused on it, Platonism is hard to maintain." (p. 42)

Only he doesn't recognize it as a solution from his materialist position. This blind spot is evident in his final comment on the subject:

> "Platonism and formalism, each in its own way, falsify part of daily experience. We talk formalism when compelled to face the mystical, antiscientific essence of Platonic idealism; we fall back to Platonism when we realize that the formalist description of mathematics has only a distant resemblance to our actual knowledge of mathematics. To abandon both, we must abandon absolute certainty, and develop a philosophy faithful to mathematical experience." (p. 43)

He presupposes that mathematicians universally have no mystical experiences in their daily lives. He tacitly assumes that any knowledge gained outside the boundaries of the scientific method is antiscientific, contrary to the position we saw Gould take.

In the last two sections of this chapter Hersh anecdotally demonstrates that mathematicians make mistakes, can sometimes have trouble understanding one another, and sometimes have insight to mathematical truth without proof. On the one hand he is just making the same error in not distinguishing between the epistemological and the ontological aspects of mathematics. Beyond that, even if you assume his perspective and ignore the error, his argument presupposes its conclusion. On the other hand all of these phenomena support the Platonic position.

After Myths/Mistakes/Misunderstandings, Hersh takes us on a tour entitled Intuition/Proof/Certainty, describing how these concepts relate to the philosophy of mathematics. This chapter is much more difficult to address briefly because of the subtle way in which healthy servings of truth are leavened with the spin of Hersh's predisposed doctrine. Throughout the chapter he is on the right track in exposing crucial problematical issues in philosophy (as they pertain to mathematics), yet rather than attempt to resolve these issues on a philosophical plane Hersh offers, by decree, their existence as "proof" of humanism's adequacy.

Hersh exposes a dichotomy in our notion of proof in order to attack the authority of mathematics. On the one hand we have an understanding of proof as a rigorously formal process of deduction. On the other hand, in practice proofs rarely, if ever, conform to this formal standard. He's right. He asks why so few mathematicians recognize this discrepancy and whether it matters. "It matters, morally, psychologically, and philosophically," he asserts (p. 49). He's right again. He's wrong, though, in thinking that this subverts the objective authority of mathematics. Unfortunately, space and the scope of this review prevent a full refutation of his interpretation. Suffice it to say that the key elements that Hersh fails to grasp include:

- Formal deduction only captures the syntactical aspects of a domain. Its semantics must be supplied by some interpretation *external* to the formal proof itself.
- Beyond Hersh's conception of a proof as a tool for explaining or convincing, proofs are constrained by some context of what is already "known," upon which they build.
- Proofs are ontologically different *in kind* from the mathematical objects to which they refer, on the one hand, and the human creative process of discovering the proof, on the other.
- Proofs only have epistemological implications, not ontological ones. Therefore, proofs that are incomplete or that contain errors have relevance to our knowledge of mathematics, not to the objective truth of mathematical reality (see his discussion of probabilistic proofs, p. 56).

The article on intuition is one of the more convoluted, confused, and self-contradictory passages in the entire volume. His claim is essentially that use of mathematical intuition is ubiquitous, but that no one really understands what it is or how it works. To demonstrate its vague and ambiguous nature, he provides a list of six meanings that the term takes on in various mathematical contexts. But rather than resolving the ambiguity by providing some scientific explication, he is content to use the ambiguity to his own advantage.

What Hersh apparently never recognizes is that intuition, as he describes it, is a phenomenon with much broader applicability than mathematics.[5] Intuition is just as ubiquitous in the sciences, in the arts, in the humanities, in engineering, in all human endeavors. Intuition is a primitive human faculty of the mind. It therefore falls under the purview of philosophy of mind, not philosophy of mathematics. However, to my knowledge, none of the cognitive sciences have ever accounted for it.

Despite this, in a bold maneuver Hersh proposes a cognitive mechanism for intuition, citing **no** references to supporting literature or research:

> "Intuition isn't direct perception of something external. It's the effect in the mind/brain of manipulating concrete objects—at a later stage, of making marks on paper, and still later, manipulating mental images. This experience leaves a trace, an effect, in the mind/brain. That trace of manipulative experience is your representation of the natural numbers. Your representation is equivalent to mine in the sense that we both give the same answer to any question you ask...We have intuition because we have mental representations of mathematical objects...Different people's representations are always being rubbed against each other to make sure they're congruent. We don't know how these representations are held in the mind/brain. We don't know how *any* thought or knowledge is held in the mind/brain." (p. 65)

Particularly bold in light of these last two admissions. Unfortunately, his model has no explanatory power other than to facilitate his own definition of mathematics (begging the question):

> "*...the study of mental objects with reproducible properties is called mathematics*. Intuition is the faculty by which we consider or examine these internal, mental objects." (p. 66)

Also offered with no supporting empirical evidence or contextual theory. He brings the argument full circle by relating this cognitive model to the issue of fallibility in mathematics.

> "There's always some discrepancy between my intuition and yours...Sometimes a question has no answer.....We know that with physical objects we may ask questions that are inappropriate, which have no answer. What are the exact velocity and position of an electron? How many trees are there growing at this moment in Minnesota?[6] For mental objects as for physical ones, what seems at first an appropriate question is sometimes discovered, perhaps with great difficulty, to be inappropriate. This doesn't contradict the existence of the particular mental or physical object. There are questions that *are* appropriate, to which reliable answers can be given" (p. 66)

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[5] While I agree wholeheartedly with his assertion that "Accounting for intuitive 'knowledge' in mathematics is the basic problem of mathematical epistemology," (p. 65) I would generalize this to the equally true assertion obtained by omitting the words 'in mathematics' and 'mathematical'.

[6] There is a difference between a question having no answer and being unanswerable. There *are* a specific number of trees growing in Minnesota at the moment you read this. It is merely infeasible to count them. The issue of an electron's position and velocity is subtly complex, requiring more precision and care than implied in the passage (see Wicks, 1999). It is not the case that these questions don't have answers, we simply are unable to know what they are. Hersh is once again confusing the ontological with the epistemological.

Rather than take the questionable pedagogical route of classifying questions as appropriate or not, it would be more intellectually satisfying to not only recognize that some questions are empirically inaccessible or epistemologically unknowable, but to strive to understand the character of this boundary between knowledge and reality.

In perhaps the clearest, most definitive expression of a mindset symptomatic of materialist, humanist ideology, Hersh opens his comments on certainty with the following stipulation and question:

> "Even if it's granted that the need for certainty is inherited from the ancient past, and is religiously motivated, its validity is independent of its history and its motivation. The question remains: is mathematical knowledge indubitable?" (p. 66)

Setting aside the historical and religious motivation, however, becomes more onerous than Hersh likely anticipated. For certainty is grounded in faith, and faith is indispensable. If, according to my premise, all knowledge is predicated on faith, then any certainty we ascribe to our knowledge must be reducible to our faith in the primitive axioms upon which that knowledge is built.

Webster defines faith as "firm belief in something for which there is no proof." In the epistle to the Hebrews the author says of faith that it is "being certain of what we do not see." (Hebrews 11:1, New International Version) Both of these are consistent with the concept that *certainty* of mathematical knowledge is derived from faith in axiomatic assumptions. In the light of this conclusion, Hersh's question is not fundamentally about the certainty of mathematical knowledge (as he would claim), but rather the commensurability of faith and doubt.

The essence of Hersh's point is that if it is conceivably possible to doubt the reliability of mathematical reasoning, proof, axiomatic assumption, etc., then certainty is obliterated. But this relies on a facile concept of the relationship between faith and doubt, one of mutual exclusivity. Much more sophisticated metaphysical views, however, have been posited and defended. For example, the novelist and theologian Frederick Buechner has this to say on the subjects of doubt and faith:

> "Whether your faith is that there is a God or that there is not a God, if you don't have any doubts you are either kidding yourself or asleep. Doubts are the ants in the pants of faith. They keep it awake and moving...
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> "Tillich said that doubt isn't the opposite of faith; it is an element of faith...
>
> "The five so-called proofs for the existence of God will never prove to unfaith that God exists. They are merely five ways of describing the existence of the God you have faith in already." (Buechner, 1973)

These principles apply just as readily to mathematics. Whether we doubt the ability of human mathematicians to get it right (due to finite ability and frailty), or doubt the applicability of mathematics to physical reality (due to incomplete models), or doubt the truth of axiomatic mathematical assumptions (due to inductive processes) this doubt has no impact on the objective certainty of mathematical ontology.

In the final chapter of Part One of Hersh's book he revisits five classical puzzles that have been the center of philosophical debate for centuries:

> "Is mathematics created or discovered? What is a mathematical object? Object versus process. What is mathematical existence? Does the infinite exist?" (p. ix)

Rather than adding anything substantive to our understanding of these issues, Hersh continues his pattern of rhetoric—"Platonists **think** mathematical entities can't be created...Formalists and intuitionists, on the contrary, **know** that mathematics is created by people." (p. 73, emphasis added)—linguistic sleight of hand—"How can I claim that 2 + 2 = 4 isn't timeless...Euclid never saw the formula 2 + 2 = 4."[7] (p. 81)—and open hostility to the beliefs of others:

> "Or is there and was there, already in Pythagoras's time, one majestic, unique, eternal 2, already an integer, already a rational number, already a real number, already a complex number, *and* who knows what else? **These are fables you can believe if you want to.**" (p. 82, emphasis added)
>
> "One can't help recalling the diktat of Ludwig Kronecker: 'The integers were created by God; all else is the work of man.' Since Kronecker was a believer, it's **possible** he meant this literally. But when mathematicians quote it nowadays, **'God' is a figure of speech.**" (p. 74, emphasis added)

In an argument consistent with his inability to correctly distinguish between concepts (e.g., representation and referent, ontology and epistemology), Hersh attempts to make a case for object and process existing at extremes of a continuum entailing social-constructivism. He suggests that the process of mathematical development, maturation, and advancement causes a fundamental change in the objects of mathematics. But this type of argument applied to biology (or any other science) would be laughable[8]. It is inconceivable that the discoveries of germs as a cause of disease, cells as the building-blocks of plant and animal life, and DNA as the substrate of inheritance and genetic identity have in any way changed the fundamental nature of living organisms on our planet (or universe). Did DNA have a helical structure before Watson and Crick? Of course. *In science and mathematics, epistemological advancement effects no ontological metamorphoses.*[9]

IV

Part Two of *What Is Mathematics, Really?* offers up a unique survey of the philosophy of mathematics by focusing on key mathematicians and philosophers within the contexts of their historical perspectives. In telling their stories Hersh is successful in attaining his goal: "...try to be entertaining and relevant, not definitive or exhaustive. Most of the facts are well known. The arrangement and some interpretation are novel." (p. 91) These seven chap-

[7] Although Euclid may have never seen the *formula* 2 + 2 = 4 (nor the formula 2 + 2 = 2 × 2), he would have been familiar with the belief of the Pythagoreans that the duad (the Platonic form from which the number two derives) was equal, meaning that "'two and two are equal to twice two:' that is, the addition of two to itself, is equal to the multiplication of it by itself." (Taylor, 1816) So in spite of having a less sophisticated representation, the concept was the same.

[8] At least I thought so, until sociologist of science Bruno Latour's article 'Ramsès II est-il mort de la tuberculose?' appeared in *La Recherche*, in which he asks how it is possible that the Egyptian pharaoh Ramsès II could have died about 1200 B.C.E. from tuberculosis when the tuberculosis bacillus wasn't discovered until 1882. (see Sokal and Bricmont (1998), pp. 96-7)

[9] Insofar as science and mathematics are objective. The scientific method can eliminate candidate ontological descriptions (models) but can not select between two proposed competing ontologies (see Wicks, 1999).

ters, while spattered with an occasional comment reflecting Hersh's own bias[10], are largely fair, objective representations of the beliefs and contributions of the historical figures.

Hersh divides his account vertically through time into two parallel tracks. First tracing the school of thought he refers to as *Mainstream* back to Plato and the Pythagoreans, he explores the relationship that evolved over the centuries between mathematics and mysticism/theology/religion. Then to explain the crisis that has occurred this century in foundationism, he shows how humanism and rationalism (legacies of Protagoras and Aristotle) led to the humanist school of thought in mathematics characterized by people he refers to as *Mavericks*. The intellectual environment of the enlightenment facilitated this dichotomy, and by the twentieth century the shift in the balance of power was complete. As he observes: "Recent troubles in philosophy of mathematics are ultimately a consequence of the banishing of religion from science." (p. 122); "The trouble with today's Platonism is that it gives up God, but wants to keep mathematics a thought in the mind of God." (p. 135)

In marvelous detail and accessible prose Hersh carefully weaves together source quotations, paraphrases of doctrines and beliefs, and discussions of over fifty mathematical thinkers, producing an unrivaled evolutionary narrative of thought on the nature of mathematics. For this reason alone, *What Is Mathematics, Really?* deserves to be listed among standard readings in mathematics curricula. It surely will serve as a reference and sourcebook for philosophical historians and mathematical anthropologists.

V

Hersh's final chapter begins with a self-assessment:

> "Chapter 2 considered a list of criteria for a philosophy of mathematics. How do we look according to our own tests? I reprint the list:
>
> 1. Breadth
> 2. Connected with epistemology and philosophy of science
> 3. Valid against practice: research, applications, teaching, history, computing, intuition
> 4. Elegance
> 5. Economy
> 6. Comprehensibility
> 7. Precision
> 8. Simplicity
> 9. Consistency
> 10. Originality/novelty
> 11. Certitude/indubitability
> 12. Acceptability" (p. 235)

The speciousness of this list, however, is due to the omission of the most fundamental criteria: "Philosophy may be defined as the art of asking the right questions." (Heschel, 1955, p. 4) Unfortunately, Hersh has asked an abundance of wrong questions (e.g., "To a formalist or Platonist who presents an inhuman picture of mathematics, I ask, 'If this were so, how could anyone learn it?'" (p. 27)).

More interesting, perhaps, than my own evaluation of his philosophy by these criteria is the unimpressive grade he gives himself. To summarize, Hersh believes he has presented a philosophy that does better than its predecessors in criteria 1-3, believes 4, 5 and 8 to be unimportant, believes 7 and 11 to be impossible or undesirable goals, and believes he gave 6, 9, and 10 the ol' college try. Acceptability (12) he left as an open question. Even if we were

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[10] The most significant of which is his persistent and open hostility to religion, e.g., "Fundamentalist religion is spreading like a noxious weed." (p. 122)

willing to stipulate to his evaluation, achieving success in only three to six out of twelve criteria does not, in my mind, constitute a victory.

Following the report card, Hersh dedicates slightly less than a full page to the implications of philosophy for teaching and pedagogy. Beyond stating the obvious, that each should have an influence on the other, he fails miserably to add anything substantive to the discussion. Throughout the book, he refers to the immaterial nature of Platonism as a barrier to education. This argument is a red herring. The problem of transfer of knowledge, experience, and insight has little to do with its fundamental nature or content, but everything to do with your philosophy of mind and its relationship to your general philosophy of education and learning.

The most irresponsible statement Hersh makes, however, relates to Platonism's effect on education:

> "Platonism can justify a student's certainty that it's impossible for her/him to understand mathematics. Platonism can justify belief that some people can't learn math. Elitism in education and Platonism in philosophy naturally fit together… [Humanist philosophy] can't do damage the way…Platonism can." (p. 238)

This assertion is made with no corroborating evidence, no defense, no case studies, and closely following this (hopefully embarrassing) confession: "I haven't seen the effect of Platonism on teaching described in print." (p. 238) It remains to be seen what sorts of damage Humanism will wreak.

Hersh ends his book with a curious, yet mildly interesting, study of the correlation throughout history between the philosophical views mathematicians held (concerning mathematics) and their political ideologies. Unfortunately, he opens this section with a misconception: "In our half of the twentieth century, it's unacceptable to import ideology into scholarship." (p. 238) I would contend that it is virtually impossible not to! What has happened during the last fifty years would be more accurately described as the ideological pendulum swinging to the other extreme insofar as scholarly acceptability is concerned. Hersh simply does not recognize, or would deny, that Humanism (in the larger sense) is an ideology that is the sustaining force behind his philosophy of mathematics.

That issue aside, the remainder of this concluding chapter can not be taken seriously, by any sensible measure, as anything other than pure rhetoric. Hersh only justifies this (possibly veridical) cooked up correlation by way of a self-serving, biased demarcation of right and left in the political sphere and a tally of sporadic historical figures that fall far short of statistical significance. He unabashedly reveals his purpose in the book's final paragraphs: The correlation is intended to establish that "The humanist philosophy of mathematics has a pedigree as venerable as that of Mainstream. Its advocates are respected thinkers." (p. 249) In other words, members of the academy who resonate with the social or political left should feel obligated to embrace the social-constructivist philosophy of mathematics out of solidarity and the credentials of its advocates, regardless of its merits.

Nevertheless, I do agree that one's philosophy of mathematics is necessarily intertwined with one's broader philosophical, political, pedagogical, religious, cognitive, and social belief structure, all of which emerge out of the primitive elements of one's faith. As such, introspective clarity is an essential prerequisite to both the practice and the education of mathematics. In reminding us of this, Hersh is to be commended.

VI

"The world of ideas which it discloses or illuminates, the contemplation of divine beauty and order which it induces, the harmonious connexion of its parts, the infinite hierarchy and absolute evidence of the truths with which it is concerned, these, and such like, are the surest grounds of the title of mathematics to human regard, and would remain unimpeached and unimpaired were the plan of the universe unrolled like a map at our feet, and the mind of man qualified to take in the whole scheme of creation at a glance.

—J. J. Sylvester, nineteenth-century mathematician"
(quoted from Osserman (1995, p. 101)

Mathematical knowledge ranks amongst the highest pinnacles of human intellectual achievement. Not because we in any way are responsible for its existence (any more than we can claim credit for establishing gravity), but rather for the depth to which we have been able to probe its boundaries and consequences, unraveling its mysteries. And we cannot deny that these mysteries exist, and even persist in the face of explanation. For intellectual endeavors often fall short of true explanation, settling instead for mere description. For example, knowledge of ontogenesis, optics, and the arithmetic of proportionality will provide a descriptive account of the ubiquitous role of the Golden Section in biology, aesthetics, and geometry. Nevertheless, I still marvel at the mystery of how a simple irrational constant is intricately connected to the very fabric of reality and life (see Ghyka, 1946). Rationalism and Humanism both fail to account for anything beyond the (relatively sterile) descriptive level of knowledge.

So to this extent I agree with Hersh that 20th century pedagogy has failed by limiting the focus to mechanistic, formal methods that fall short of providing deep conceptual understanding. It is the lack of attention to the divine, aesthetic, mystical properties, and *otherness* of mathematics that has led to such modern phenomena as innumeracy and math phobia. To conceive of mathematics as cold and perfunctory is to hate it. Unlike Hersh, however, I stop short of the hubris required to believe that all mystery vanishes at the hand of Man. Without reference to the supernatural, the divine, a Creator, there is no foundation for knowledge. Reason cannot bootstrap itself. "The worship of reason is arrogance and betrays a lack of intelligence. The rejection of reason is cowardice and betrays a lack of faith." (Heschel, 1955, p. 20)

The recent publications of *Fashionable Nonsense* (Sokal and Bricmont, 1998) and *A House Built On Sand* (Koertge, 1998) provide well-documented evidence that Hersh is not alone in his efforts to extend the influence of social-constructivism from psychology, sociology, and literary criticism to the natural sciences and mathematics. My concern is that he appears to be one of the first *mathematicians* to do so. Sokal and Bricmont are able to speculate as to how non-scientists could misrepresent and conflate subtle concepts in mathematics and science (e.g., shallow understanding of the concepts; confusion between popular and technical meanings of words; inappropriate generalization or transfer of terms to other disciplines; etc.). Given Hersh's credentials as a career mathematician and educator, I can not as kindly give him the benefit of the doubt.

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References

Barth, Paul (1959), *Dogmatics in Outline* (New York: Harper & Row).

Buechner, Frederick (1973), *Wishful Thinking: A Theological ABC* (San Francisco: HarperCollins).

Ernest, Paul (1991), *The Philosophy of Mathematics Education* (New York: Falmer).

Ernest, Paul (1997), *Social Constructivism in the Philosophy of Mathematics* (Albany: SUNY Press).

Gardner, Martin (1996), 'Computers near the threshold?', *Journal of Consciousness Studies*, 3 (1), pp. 89-94.

Ghyka, Matila (1946, 1977), *The Geometry of Art and Life* (New York: Dover Publications, Inc.).

Gould, Stephen Jay (1998), 'In Gratuitous Battle', *Civilization*, October/November, pp. 86-88.

Heschel, Abraham Joshua (1955), *God in Search of Man: A Philosophy of Judaism*, (New York: Farrar, Straus and Giroux).

Hersh, Reuben (1997), *What Is Mathematics, Really?* (New York: Oxford University Press).

Horgan, John (1996), *The End of Science: Facing the Limits of Knowledge in the Twilight of the Scientific Age* (New York: Broadway Books).

Koertge, Noretta [Editor] (1998), *A House Built on Sand: Exposing Postmodernist Myths About Science* (New York: Oxford University Press).

Osserman, Robert (1995), *Poetry of the Universe: A Mathematical Exploration of the Cosmos* (New York: Doubleday).

Snow, C.P. (1969), *The Two Cultures and A Second Look* (London: Cambridge University Press).

Sokal, Alan, and Bricmont, Jean (1998), *Fashionable Nonsense: Postmodern Intellectuals' Abuse of Science* (New York: Picador).

Taylor, Thomas (1816), *The Theoretic Arithmetic of the Pythagoreans*. Reprinted in 1972 by Samuel Weiser, Inc., New York.

Wicks, John (1999), 'Determinism, Chaos, and the Meaning of Randomness, or Quantum Theory and the Sovereignty of God,' in Brabenec, Robert L., editor, *Proceedings of the Twelfth Conference of the Association of Christians in the Mathematical Sciences* (Wheaton, Illinois: ACMS).

Wilson, E.O. (1998), *Consilience*, (New York: Alfred A. Knopf).