A Greater Tantalizer

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The classic puzzle known originally as the *Great Tantalizer* and more recently as *Instant Insanity* consists of four cubes each of whose faces have been colored with one of four colors. The object is to stack the cubes so that on each stack face all four colors are displayed. This game can be played with the other Platonic solids as well. In particular, we consider an octahedral version, consisting of six octahedra each of whose faces have been colored with one of six colors, the object being to stack the octahedra so that on each of the six natural stack faces all six colors are displayed. Whereas the color configuration in the cubic game has been arranged herein so that there are three solutions out of 41,472 possible arrangements, the color configuration in the octahedral game has been arranged so that there is but one solution out of 318,504,960 possible arrangements, which means that unless one has incredible luck or intuition or a means of focussing on the salient structure of the puzzle, solving the puzzle is practically impossible. A little bit of graph theory is just what is needed to transform these nearly intractable-by-trial-and-error combinatorial puzzles into comparatively easy puzzles.

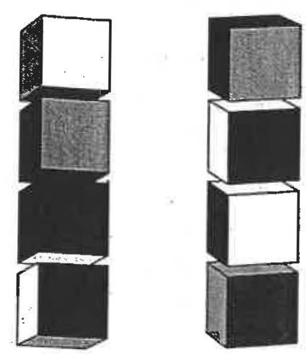

FIGURE 1. Cubic stack, front and rear view

A cubic tantalizer

We first consider a cubic version of the Great Tantalizer, modifying the analysis due to O'Beirne [3], which is duplicated in [1], [4], and [5]. Imagine having four cubes, each of whose faces have been colored solidly with R = red, W = white, G = green or B = blue. Since this paper is rendered in shades of gray, we identify B with solid *b*lack, R with dark gray, G with light *g*ray, and W with *w*hite. Note that the stack of FIGURE 1 gives a solution to the puzzle for the four given blocks. That is, each of the four *stack faces*—the faces of the stack—display all four colors. For this particular coloring of four cubes, there happens to be three distinct solutions. Finding these solutions (including the one shown in FIGURE 1) is a clever application of graph theory. We use a stereographic projection of the cube as depicted in FIGURE 2 as suggested by Richard Guy in a private communication, so that we can see all 6 faces of the cube at once rather than just 3 faces; note that the top face is rendered as a small square, the four side faces as trapezoids, and the bottom face as an unbounded region. The labels *front, left, back,* and *right* are taken with respect to the perspective of standing to the left of the cube of FIGURE 2. For ease of presentation in this paper, we shall stack the blocks horizontally across the page rather than vertically.

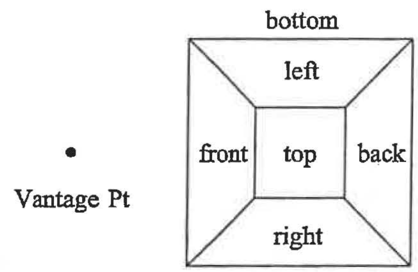

FIGURE 2. Stereographic layout of the cube

FIGURE 3 shows the color coding of the four cubes in stereographic fashion, labeled blocks *I, II, III,* and *IV.* We wish to tumble these four blocks so as to achieve the solution as given in FIGURE 1.

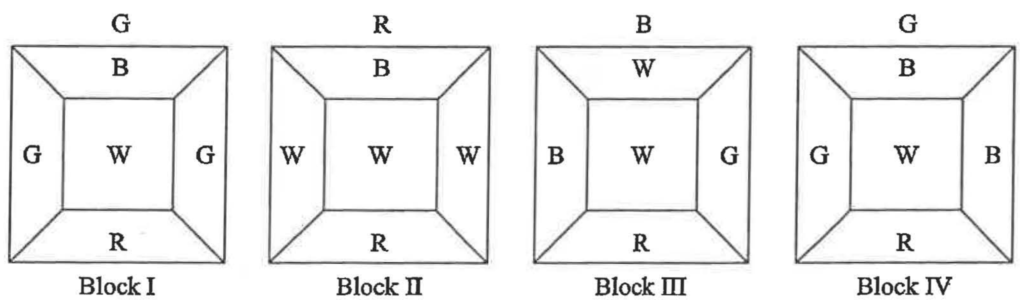

FIGURE 3. Colorings of the four cubes

The key to finding such solutions to the puzzle is the simple observation that if a particular face F of a cube C belongs to a stack face of a solution then the face opposite F on cube C also belongs to a stack face of that solution. To focus on this feature we construct the *multigraph* $\mathcal{G}$ whose vertices are the four colors R, W, G, and B, and whose edges are the color pairs of opposite faces on the cubes. Since each cube has three pair of opposite faces, $\mathcal{G}$ has 12 edges, as shown in FIGURE 4. Each edge bears the label corresponding to the cube number associated with the color pair. For simplicity, multiple edges between vertices are indicated with multiple labels, rather than multiple curves. Thus, the edge between vertices R and B is labeled 124, indicating that blocks *I, II,* and *IV* all have a pair of opposite faces, one of which is colored red and the other is colored blue. Note that blocks *I* and *II* each have a pair of opposite faces both of which are the same color; as a result $\mathcal{G}$ has two trivial edges, one connecting G with itself, and one connecting W with itself.

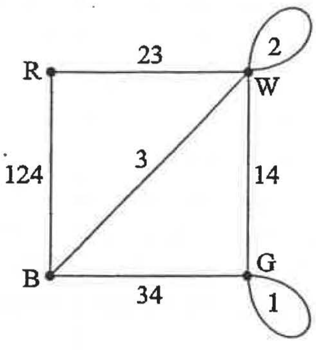

FIGURE 4. The multigraph $\mathcal{G}$

If we group all of the front and back face pairs of the cubes in a solution stack, writing them in the graph theoretic format as above, the result is a subgraph of $\mathcal{G}$ consisting of $\mathcal{G}$'s four vertices and four of $\mathcal{G}$'s edges. Furthermore, this subgraph will be a set of disjoint cycles! Therefore, to find a solution to this puzzle of stacking four blocks, we content ourselves with finding ways of decomposing $\mathcal{G}$ into two subgraphs, each of which subgraph contains all the vertices and is a set of disjoint cycles. One of these *complete sets of disjoint cycles* will correspond to the face colors on the front and back stack faces, and the other will correspond to the face colors on the left and right stack faces.

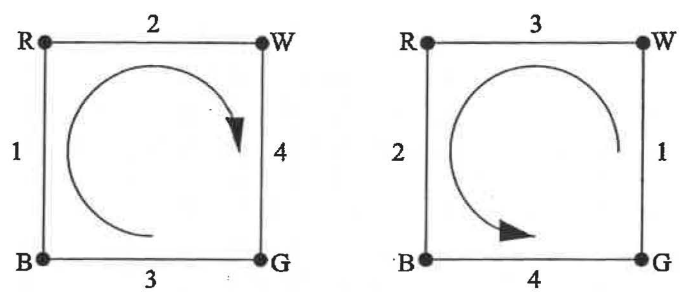

FIGURE 5. A decomposition of $\mathcal{G}$

For the multigraph of FIGURE 4 there are only two such decompositions. One decomposition is the set of two cycles which we write as (2431) (3142), and which we interpret as being the sequence of edge labels between the vertices *RWGB*, as indicated in FIGURE 5. To utilize this decomposition to stack the four blocks, let us arbitrarily assign the first cycle to correspond to the colorings of the left and right stack faces, and the second cycle to correspond to the colorings of the front and back stack faces. To build the solution stack, pick up block *I*; find the edge labeled 1 on the first cycle; observe that this edge connects *R* and *B*; arbitrarily choose one of these colors to be on the left face and tumble block *I* appropriately so that these two colors are on the left and right of the block. In so doing, this subgraph has become a *directed* graph. In FIGURE 5, note that the first cycle is directed clockwise, indicating that *B* is on the left and *R* is on the right of block *I*. Before setting block *I* down, find the edge labeled 1 on the second cycle; observe that this edge connects *W* and *G*; arbitrarily choose one of these colors to be on the front and back, so orienting the cycle, and rotate block I appropriately so that these two colors are indeed on the front and back of the block. Then set block I down. In FIGURE 5, since the second cycle is directed counterclockwise, then G is on the front and W is on the back of block I. Now consider block II; from FIGURE 5 the two cycles indicate that the front and back faces should be R and B, respectively, and that the left and right faces should be R and W, respectively. Tumble block II so as to achieve this positioning of colors and set it down on block I. Continue the process, so achieving the stereographic stack solution as given in FIGURE 6, which is equivalent to the solution as illustrated in FIGURE 1.

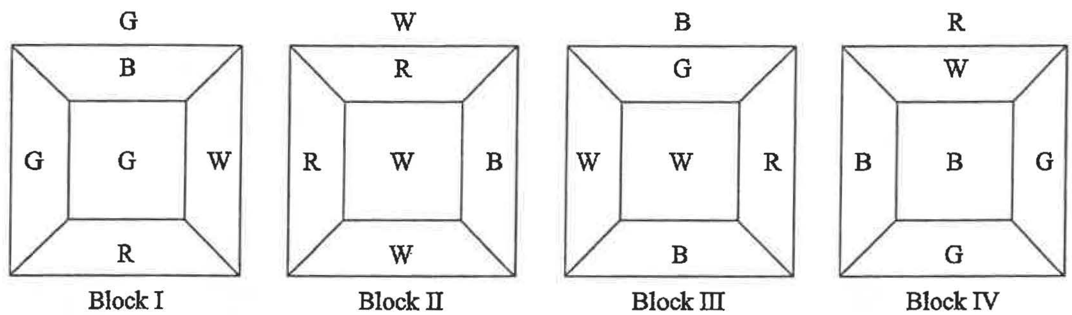

FIGURE 6. A stereographic solution stack

The two other solutions for this puzzle correspond to $\mathcal{G}$'s only other decomposition, which we write as (1)(234) (3142). The first four numbers correspond to the subgraph of two cycles, one of which is the singleton cycle G whose sole edge is a trivial edge corresponding to the color pair G/G from block I, and the other of which is the cycle between the vertices RWB; the second four numbers correspond to the cycle between the vertices $RWGB$. FIGURE 7 illustrates this decomposition, as well as an orientation. The reader may wish to build the corresponding stereographic solution stack, similar to that displayed in FIGURE 6. Note that in a solution stack corresponding to the decomposition of FIGURE 7, block I can be rotated giving a (marginally) different stacking solution, the only difference being that the top and bottom faces of block I have switched positions.

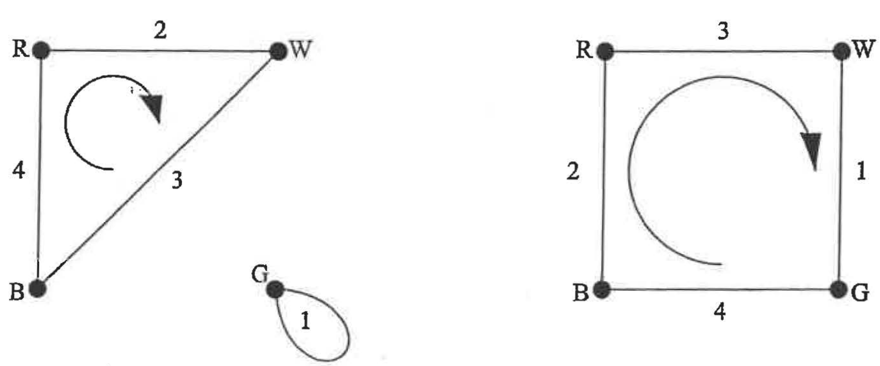

FIGURE 7. A second decomposition of $\mathcal{G}$

Finally, to count the number of distinct ways of stacking the blocks without regard to order, arbitrarily take block I as the cornerstone; since there are three pair of opposite faces, select one of these three as being the top/bottom faces. (Note that a stack upside down is really the same stack.) Set block I down on one of these two faces. Each of the remaining blocks can be placed on the stack on any of their 6 faces, and rotated 4 ways, which means that the total number of ways of stacking the four blocks is $3 \cdot 24^3 = 41,472$ different ways.

An octahedral tantalizer

Now let's play the same game with octahedral blocks. This time we use six blocks, and six colors. FIGURE 8(A) shows a shaded stack face for a set of six octahedra. If we use the colors R = red, W = white, B = blue, E = ebony, Y = yellow, and G = green, and if we think of E as being solid black, G as being darkish gray, R as lightish gray, B as off-black (between E and G), and Y as off-white (between R and W), then FIGURE 8(B) and 8(C) give a color coding and shaded puzzle solution, respectively, for a given set of six blocks.

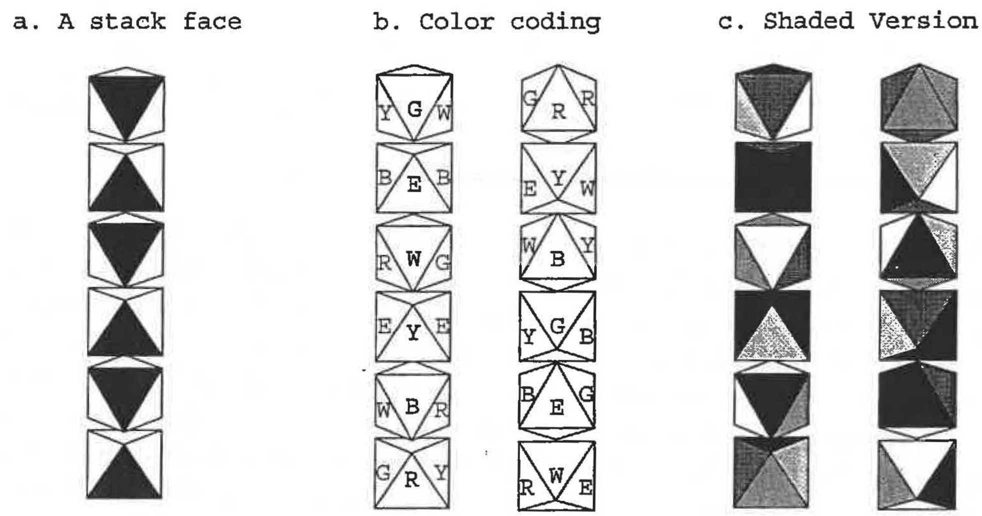

FIGURE 8. Octahedral stacks

Once again, it is helpful to use a stereographic projection of the octahedron. In particular, let's consider six blocks whose faces are colored according to the scheme of FIGURE 9. For example, note that block I is colored so that its top face is W, its bottom face is R, and whose side faces are colored G, E, W, R, Y, and B, in a clockwise circuit around the octahedron. Note that each block has been also rendered as a *dual* graph, wherein each face of the octahedron has collapsed to a vertex labeled with the face's color, with an edge between vertices if and only if the corresponding faces of the octahedron share an edge. Observe that different vertices in these dual graphs may in fact have the same vertex label. The dual graphs harbor readily accessible information about face adjacencies in the octahedral blocks, information which will be needed later on in our analysis.

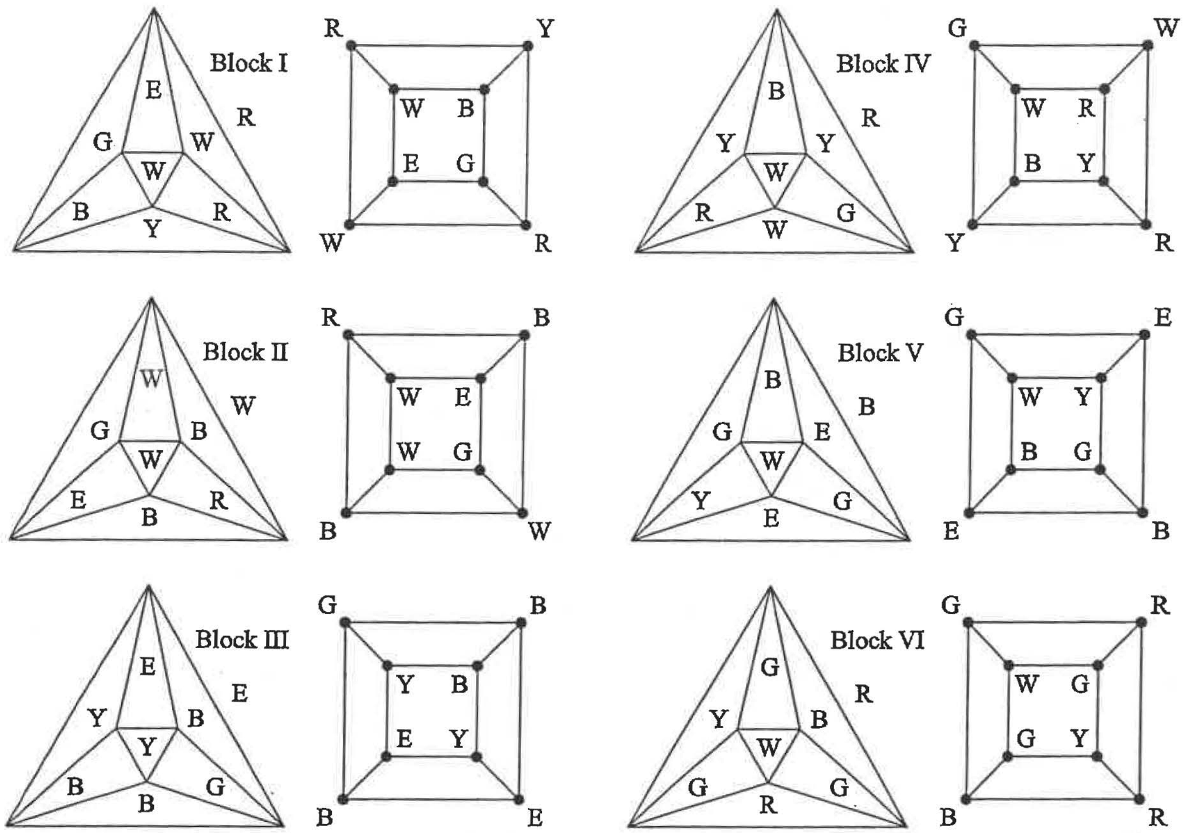

FIGURE 9. Color coding of the octahedra

Next we form the multigraph $\mathcal{G}$ as in FIGURE 10, which has six vertices, and whose edges correspond to the color pairs on opposite faces of the six octahedra.

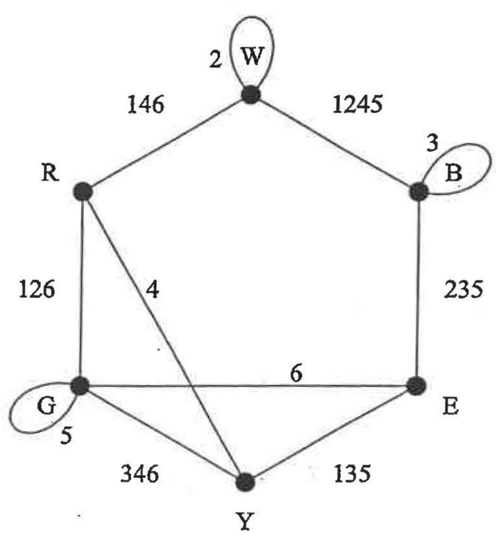

FIGURE 10. The color-pair multigraph $\mathcal{G}$

As before, we decompose $\mathcal{G}$. But this time a decomposition will consist of three complete sets of disjoint cycles corresponding to the left-and-its-rear stack face, the front-and-its-rear stack face, and the right-and-its-rear stack face. Such decompositions we call *treasure maps* to the puzzle, and each of the three complete sets of disjoint cycles of a treasure map are called *components* of the map. A *component structure* is a listing of the cycles of a component without regard to edge labels. From FIGURE 10, there are four possible component structures: *(RWBEYG)*, *(RWBEGY)*, *(W)(B)(RYEG)* and *(G)(RWBEY)* A branch and prune search through $\mathcal{G}$ turns up 17 treasure maps involving these component structures. We categorize these maps into four cases.

Case I: Each of the three treasure map components are *(RWBEYG)*. Each row entry in TABLE 1 contains three sequences of six digits which correspond to paths through the vertex sequence *RWBEYG*. For example, the sequence 123546 corresponds to the nondirected cycle $R \overset{1}{-} W \overset{2}{-} B \overset{3}{-} E \overset{5}{-} Y \overset{4}{-} G \overset{6}{-} R$.

	left: *(RWBEYG)*	front: *(RWBEYG)*	right: *(RWBEYG)*
	123546	452361	645132
	125346	453162	642531
*	142536	425361	653142
	142536	453162	625341
	143562	425136	652341
	143562	452136	625341
	145362	452136	623541
	152346	423561	645132

TABLE 1. Maps of three *straight* cycles

Case II: One treasure map component is *(RWBEGY)* and the other two components are *(RWBEYG)*. Note that if edge *EG* from $\mathcal{G}$ is used in a component, then so must edge *YR* which in turn means that such a component is *(RWBEGY)*; so in any potential treasure map containing the component *(RWBEGY)* the other two map components must be *(RWBEYG)*.

left: *(RWBEGY)*	front: *(RWBEYG)*	right: *(RWBEYG)*
125634	413562	652341
	452361	613542
152634	413562	625341
	415362	623541
	423561	615342
	425361	613542

TABLE 2. Maps of one *twisted* cycle and two *straight* cycles

Case III: One treasure map component is *(W)(B)(RYEG)* and the other two components are *(RWBEYG)*. Note that if a component contains either the cycle W or the cycle B, then it must be the set of cycles *(W)(B)(RYEG)*, and the cycle through *RYEG* must be $R \overset{4}{-} Y \overset{5}{-} E \overset{6}{-} G \overset{1}{-} R$.

left: *(W)(B)(RYEG)*	front: *(RWBEYG)*	right: *(RWBEYG)*
2, 3, 4561	145362	452136
	152346	645132
	452136	615342

TABLE 3. Maps of one *fragmented* set of cycles and two *straight* cycles

Case IV: No treasure map exists with a component of *(G)(RWBEY)*. There simply are not enough edges to form two other compatible components.

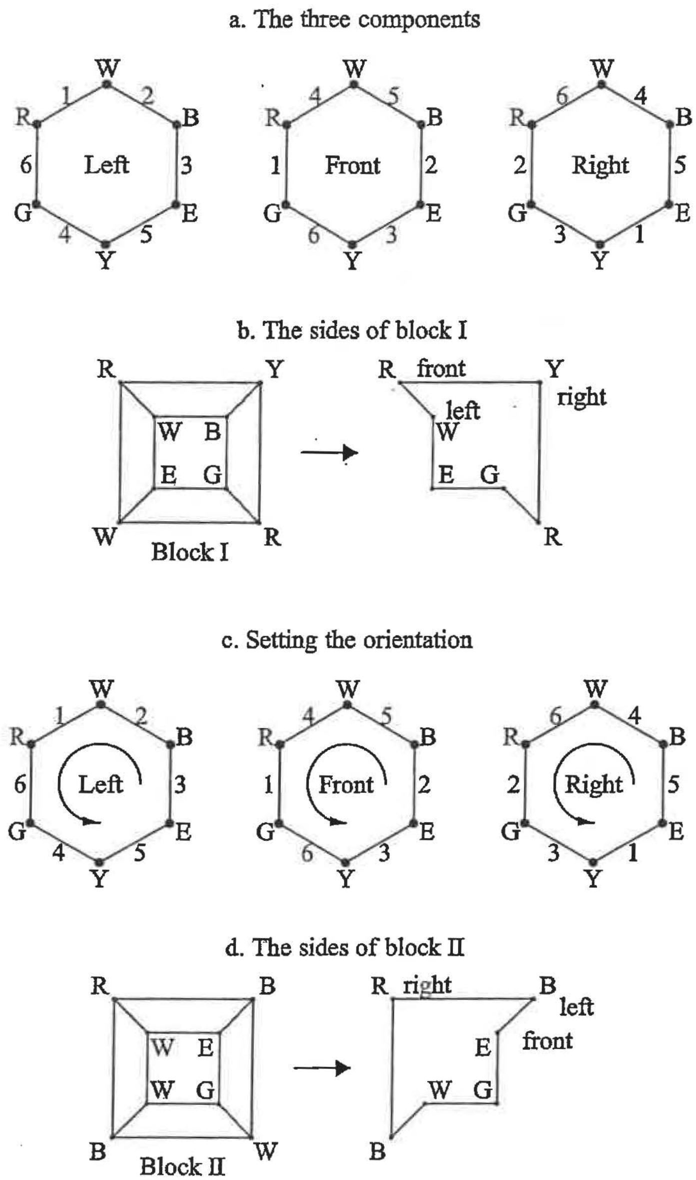

FIGURE 11. Analyzing a map

All but one of these 17 treasure maps are bogus. Finding the correct one is a process of elimination. By the face-adjacency information in FIGURE 9 which was ignored in making FIGURE 10, 16 maps lead to impossible constructions for our set of blocks. To illustrate, consider the first treasure map of Case *I*.

$$\text{left: } R \overset{1}{-} W \overset{2}{-} B \overset{3}{-} E \overset{5}{-} Y \overset{4}{-} G \overset{6}{-} R$$
$$\text{front: } R \overset{4}{-} W \overset{5}{-} B \, 2 \, E \overset{3}{-} Y \overset{6}{-} G \overset{1}{-} R$$
$$\text{center: } R \overset{6}{-} W \overset{4}{-} B \overset{5}{-} E \overset{1}{-} Y \overset{3}{-} G \overset{2}{-} R$$

We have arbitrarily chosen the first cycle to correspond to the left-and-its-rear stack faces, the second cycle to correspond to the front-and-its-rear stack faces, and the third cycle to correspond to the right-and-its-rear stack faces. Note that FIGURE 11(A) displays the graphs of these three cycles. Let's set the first block down and so transform these cycles into directed ones. To do so take the dual graph of block *I* as found in FIGURE 9, which we reproduce in FIGURE 11(B). FIGURE 11(A) tells us that the opposite face color pairs being used on block *I* are R/W on the left, R/G on the front, and E/Y on the right. From the dual graph remove the two vertices corresponding to the one color pair not being used, namely W/B, which reduces the dual graph to the cycle *RWEGRY* as shown in FIGURE 11(B); this sequence of colors is the one we should see as we walk around our block after it is set down. Note that if we arbitrarily choose the face labeled R (rather than G) as the front face, then W must be the color of the face on the left and Y must be the color of the face on the right by FIGURE 11(B). Such a relationship sets the orientation of the three cycles as shown in FIGURE 11(C), and we set block *I* down accordingly, so that R is on the front face, W is on the left face and Y is on the right. From the dual of block *II* remove the two vertices corresponding to the one color pair which is not utilized, namely W/W; the result is the cycle *RBEGWB;* with respect to the orientation as set by FIGURE 11(C), three of the vertices have been labeled *left*, *front* and *right* as shown in FIGURE 11(D); but the vertices labeled *front* and *right* are nonadjacent vertices, which means that this map is bogus.

The only treasure map to pass the test as delineated above is the one marked with an asterisk in TABLE 1. Following this map gives the directed set of three graphs of FIGURE 12, and following the directions of this set of three cycles gives a solution.

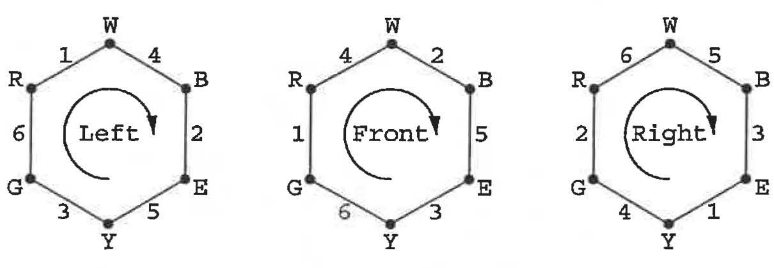

FIGURE 12. The solution orientation

In particular, set block *I* down so that its front face is G, its left is R, and its right is E. Then set block *II* down so that W is its front, B is its left, and G is its right. Continue, so constructing the solution of FIGURE 13. For simplicity the labels of the top and bottom faces have been suppressed. It should be observed that just because a treasure map passes this last test, it is still possible that such a map is bogus; in order for the solution stack to exist, the blocks must separate into two equivalence classes each containing three blocks, for in any stack face, three of the block faces of that stack must rest on one of their edges, and three must rest on one of their vertices, as can be seen from FIGURE 8. Fortunately, the asterisked treasure map passes this ultimate test.

Lastly, to compute the number of ways of stacking these 6 octahedra without regard to order, recall that in any stack the blocks are partitioned naturally into two equivalence classes of three members each. So arbitrarily select block I as the cornerstone, and select two of the remaining five blocks to join block I so as to form an equivalence class; there are 10 ways of doing this. Since each block has 4 pair of opposite face colors, select one of these to be the top and bottom face for block I; set block I down on one of these two faces. Each successive block can be set down on any of its 8 faces and rotated in any of 3 ways. Therefore the number of ways of stacking an octahedral tower of 6 blocks is $10 \cdot 4 \cdot 24^5 = 318,504,960$.

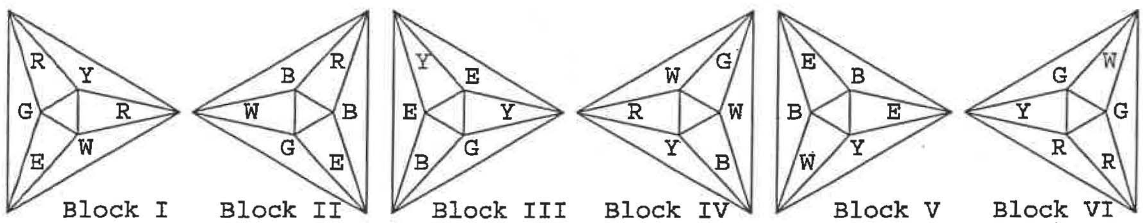

FIGURE 13. The solution

Final remarks

Great tantalizer puzzles can also be formed using tetrahedra, dodecahedra, or icosahedra, as detailed in [2]. It should be pointed out that puzzles of n-octahedra involving n colors can be solved in the way presented herein. The original puzzle as studied by O'Beirne actually involved 5 cubes with 5 different flags (rather than colors). For our octahedral tantalizer, n was chosen as 6 because there are 6 stack faces in a tower of octahedra and because 6 colors are enough to color the eight faces of each block with sufficient variety. Readers interested in obtaining colored, plastic sets of six octahedra can e-mail andrews@jebasingh.com. An applet to stack six octahedra in stereographic form is at http://www.king.edu/faculty/asimoson/Motion.htm. Finally, when invited to speak to mathematics clubs, student groups or any group interested in general problem solving, and when looking for a topic which is both readily understood and dramatically portrays the problem-solving power of mathematics, this puzzle of the Great Tantalizer is a dandy.

References.

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