THE INTERMEDIATE VALUE THEOREM

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The Intermediate Value Theorem (a continuous function on an interval assumes all values between any two of its values) is one of the big theorems of calculus. Yet the popular new Harvard book (*Calculus*, Hughes-Hallet, et al., Wiley, 1994) fails to mention it. Although most other calculus books state the theorem, they rarely use it except to show the obvious consequence: if a polynomial p satisfies $p(a)p(b) < 0$, then p has a zero between a and b. The theorem deserves better as we intend to show by listing ten picturesque consequences that we think could enliven any calculus course.

The proof of each of these results involves the construction of a function that we assume to be continuous. Continuity is more or less obvious though it may involve technicalities that we wish to avoid. You may consult the book [Boas, pp. 84-97] and its references if you wish to address this issue.

Consequence 1 (The contracting string). Imagine an elastic string to be stretched along the x-axis from $x = a$ to $x = b$. Allow the string to contract in any manner whatever so that it eventually rests along the x-axis between $x = a$ and $x = b$. There is a point of the string that ends where it started.

Proof. To see this, refer to Figure 1 and let $f(x)$ be the final coordinate of the point that started with coordinate x. Surely f is continuous (points that start close together end up close together) and so is g defined by $g(x) = f(x) - x$. But $g(a) \geq 0$ and $g(b) \leq 0$. Thus, by the IVT there is a point c such that $g(c) = 0$, that is, such that $f(c) = c$.

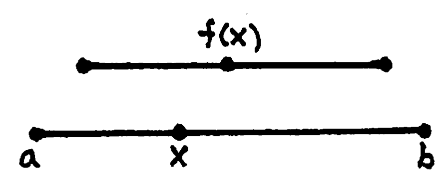

FIGURE 1

Consequence 2 (The peripatetic monk). The path wound around the mountain to the shrine at the top. Leaving his home in the foothills at 6:00 AM, a monk walked slowly along the path, often stopping to look at the flowers, and arrived at the shrine just before dark. Leaving the shrine at 6:00 AM the next morning, he hiked down the same path arriving at his home at noon. At one point on the path, his (24-hour) watch showed the same time both days.

Proof. Let the path from the monk's home to the shrine be coordinatized by x so that x = a at the home and x = b at the shrine (Figure 2). Let f(x) be the watch time at x on the first day minus the watch time at x on the second day. Note that $f(a) < 0$ whereas $f(b) > 0$. The IVT guarantees that there is a point c on the path such that $f(c) = 0$.

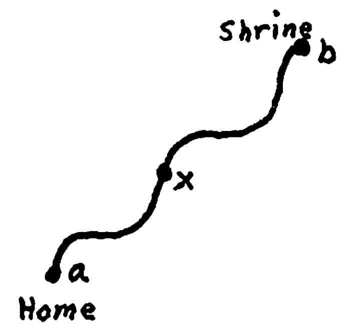

FIGURE 2

Consequence 3 (The heated ring). Introduce an uneven temperature distribution on a circular wire ring by heating it at random spots. There are always two antipodal points (points at opposite ends of a diameter) with the same temperature.

Proof. Refer to Figure 3 which has the ring in the coordinate system with center at the origin and consider a diameter with ends marked A and B. Let α denote the radian measure of the angle between the A-end of this diameter and the positive x-axis and let $g(\alpha)$ be the temperature at A minus the temperature at B. Now g(0) and $g(\pi)$ have the same magnitude but opposite signs since $\emptyset = 0$ and $\emptyset = \pi$ correspond to interchanging the position of the two ends. Thus, by the IVT there is an angle $\emptyset$ between 0 and π such that $g(\emptyset) = 0$, that is, such that the temperature at A equals the temperature at B.

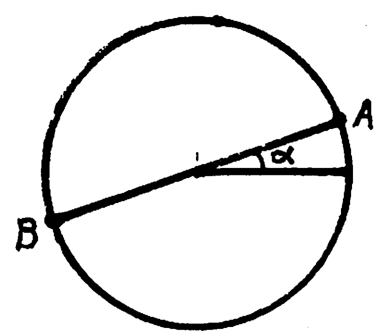

FIGURE 3

Consequence 4 (Bisecting pancakes). A Swedish pancake (a connected region in the plane) can be divided into two pieces of equal area by a single straight cut in any specified direction. In fact, two such pancakes can be bisected simultaneouly by a single straight cut (Figure 4).

Proof. See [Boas, pp. 95-96] or [Courant/Robbins, pp. 317-318].

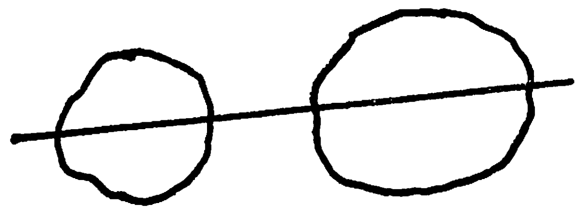

FIGURE 4

Consequence 5 (The irregular pizza). A thin pizza of any shape can be divided into four parts of equal area by two perpendicular straight cuts (Figure 5).

Proof. [Courant/Robbins, pp. 318-319].

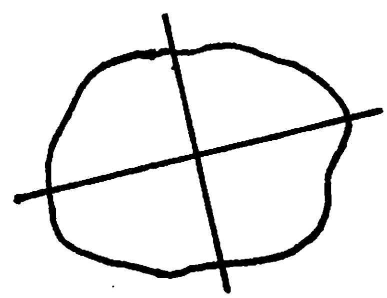

FIGURE 5

Consequence 6 (Squaring a lake). A lake of irregular shape can always be protected by a fence in the shape of a square which touches the lake on all four sides.

Proof. Refer to Figure 6 and let $f(\alpha) = A(\alpha) - B(\alpha)$. Note that $f(0)$ and $f(\pi/2)$ are equal in magnitude but of opposite sign. It follows that there is an angle $\emptyset$ such that $f(\emptyset) = A(\emptyset) - B(\emptyset) = 0$.

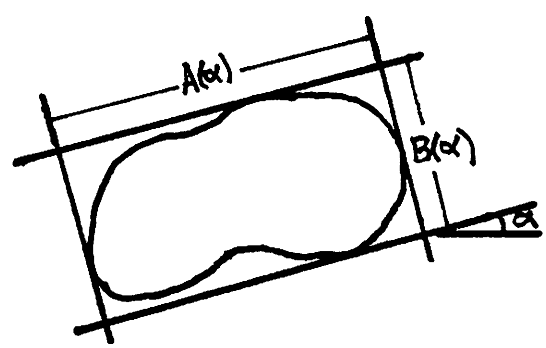

FIGURE 6

Consequence 7 (The wobbly table). Who hasn't been irked by the round (or square) four-legged table that wobbles and spills the coffee? No matter how uneven the floor (provided it's a continuous surface with modest-sized ripples), the wobble can be eliminated by an appropriate rotation of the table about its center.

Proof. Introduce a coordinate system so the center of the table is at the origin and so that legs C and D touch the floor when $\alpha = 0$ (Figure 7). Rotate the table through 90° always keeping at least one of the pairs A,B or C,D on the floor and ending with the pair A,B there. Let $A(\alpha)$ be the sum of the distances of legs A and B from the floor and let $C(\alpha)$ have a similar meaning for legs C and D. Set $f(\alpha) = A(\alpha) - C(\alpha)$ and note that $f(0) = A(0) \geq 0$ whereas $f(\pi/2) = -C(\pi/2) = -A(0) \leq 0$. We conclude that there is an angle $\emptyset$ such that $f(\emptyset) = A(\emptyset) - C(\emptyset) = 0$. But since one of $A(\alpha)$ or $C(\alpha)$ is always 0, $A(\emptyset) = C(\emptyset) = 0$.

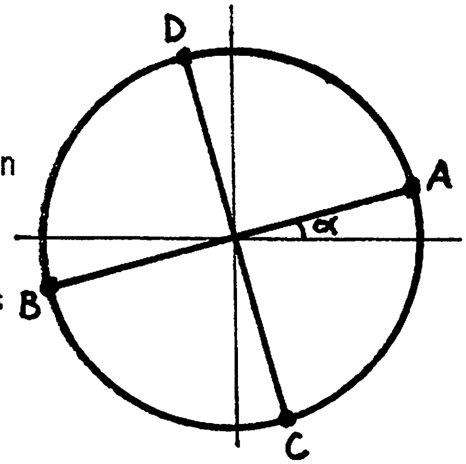

FIGURE 7

Consequence 8 (The erratic runner). A runner who paced himself very unevenly still managed to run 5 miles in exactly 40 minutes. Somewhere along the course, he ran a mile in exactly 8 minutes.

Proof. Let t measure the number of miles from the starting point and let $g(t)$ denote the time for the runner to get from point t to point t + 1. Then

$$g(0) + g(1) + g(2) + g(3) + g(4) = 40$$

From this, we infer that not all of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$ are greater than 8; nor are all of them less than 8. Thus, there are points a and b between 0 and 4 such that $g(a) \le 8 \le g(b)$. By the IVT, we conclude that there is a number c such that $g(c) = 8$; it takes 8 minutes for the runner to get from c to c + 1.

Consequence 9 (Universal chord theorem). Suppose that the continuous function f satisfies $f(0) = f(1)$. Then the graph of f has horizontal chords (endpoints on the graph) of length 1, 1/2, 1/3, . . . but may not have chords of other specified lengths (Figure 8).

Proof. [Boas, pp. 92-93]

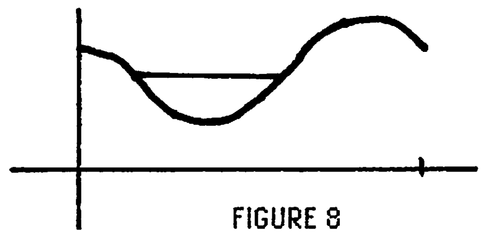

FIGURE 8

Consequence 10 (The hinged lever). The train travels from station A to station B, speeding up, slowing down, stopping occasionally, and even backing up once or twice. On the floor of one of the cars is a hinged lever (Figure 9) which may freely rotate forward or backward under the influence of gravity and the motion of the train but which once it hits the floor stays there. There is an initial angle for the lever which insures that it never hits the floor.

Proof. [Courant/Robbins, pp. 319-321].

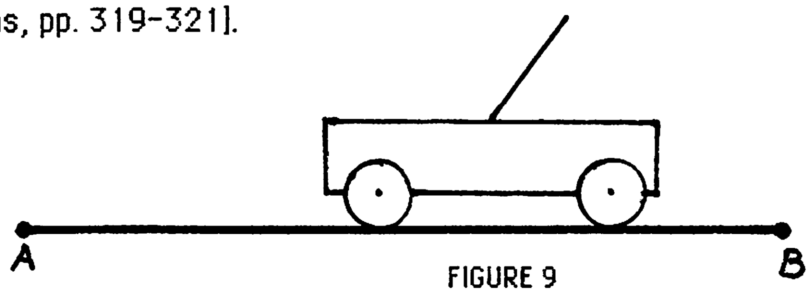

FIGURE 9

References.

Ralph P. Boas, Jr., *A primer of real functions*, 3rd edition, Mathematical Association of America, 1981.

Richard Courant and Herbert Robbins, *What is mathematics?* Oxford U. Press, 1941.