Teach a Course in the Math of Voting and Choice

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Many mathematics instructors at the college level are looking for a curricular option that has the potential to serve a number of different constituencies. It could be to encourage more students to take math courses, or to give worthwhile options to students who need to take math but who are not ready for calculus (or its sequence). On the other hand, one may wish to add a new course for majors outside of the typical offerings, or even to prepare students for undergraduate research. The mathematics of voting and choice is ideally suited to meet all these needs in the collegiate math classroom.

What sort of questions would one try to answer in such a course? The most important goal, perhaps, is the attempt to explain the many paradoxes which seem to occur in voting systems we are all familiar with. For instance, in our standard plurality voting, it happens on occasion that the person elected is in fact the least-desired option of a large majority of the populace. Related to this is the characterization and classification of various voting systems; this is an especially mathematical goal.

A more applied direction would be to search for robust applications of principles of choice discovered by these means in a wide range of social and managerial sciences – how should we make choices in those arenas, given competing interests? On the other hand, although statistics, polling, and value judgments about specific elections are certainly closely related to this area and interact with it, in the math of voting and choice we do not typically address these issues head on.

One reason the subject is a good choice for many different constituencies is that it is a relatively new and accessible field. Of course, there was activity before the last century. Medieval scholars already began discussing different systems still fought over today; in the same era, the papal election process was changed to require one candidate to earn a two-thirds majority. Before the French Revolution, many of the same issues were raised again (and more systematically) by various scientifically-minded nobility in the Academie, and an early contributor in the English-speaking world was none other than Charles Dodgson (Lewis Carroll), who proposed several interesting new methods.

But it was in the 1950s that the subject came of age. Economists grappling with how societies or companies make decisions based on many inputs came to some remarkable conclusions. The most famous of these is Kenneth Arrow's theorem that if there are three or more candidates, there is no election procedure which will always satisfy five very reasonable properties (for instance, that it not be a dictatorship, and that anyone can vote for any candidate). Another Nobel laureate, Amartya Sen, made a similar discovery around the same time, and over the next twenty years economists (largely influenced by game theory as developed by von Neumann, Morgenstern, and Nash) developed the field.

By the early 1980s, outside interests had made fundamental contributions. Most well known is the introduction by Steven Brams, a New York University political scientist, (and others) of the method called approval voting. Legislative apportionment, which had been addressed cyclically whenever controversy arose over how many seats states received after a census, was given a detailed treatment by Balinski and Young. A more axiomatic and geometric analysis of nearly all voting issues was begun slightly later by Don Saari and many others.

Today, with all the older principals still actively researching, younger researchers are expanding horizons in other directions, including a plenary talk at this conference!

So what has prompted all this activity? In a nutshell, people often feel that the candidate elected is not the one the voters 'really want', regardless of their personal preference or outcome. Indeed, many different systems in use often disagree on outcome! The simple plurality vote will often have a different result than a 'ranking' system similar to that used in collegiate football polls, for instance, and these could both disagree with a system choosing the candidate who prevails in the most head-to-head contests with other candidates. Not to mention so-called agenda elections, and many other possibilities…

Thus, mathematicians simply analyze all the methods, or at least large classes of them where practicable. At the same time, we attempt to explain real-life controversies, such as the 2000 presidential ballot in the state of Florida. A mathematician may well contend that the controversy was, at the core, due neither to poor voting equipment, a confusing ballot, or disenfranchisement, but our use of the plurality method to award the electoral votes in each state.

The sort of information we look at is represented in a *profile*, which lists the types of possible voters and the number of each in the election. For example, the profile

3 A>B>C 4 A>C>B 6 B>C>A

would indicate that while seven of thirteen voters rank candidate A first, only three of those prefer candidate B to candidate C. One may note that A wins a plurality vote, while if we focus on which candidate is least disliked (ranked last by the fewest voters), C would win, and by assigning 2 points to a voter's top candidate and 1 point to the middle one, B would win (by a score of 15, over 14 for A and 10 for C). This sort of behavior is not at all unusual. Analyzing likely profiles for the 1860 presidential election, for example, leads one to note that there are justifiable arguments for saying that Douglas, not Lincoln, was the people's 'true' choice.

However, our goal is not just to look at paradoxes; we also wish to classify. For instance, which procedures ensure that if a candidate is the first choice of a majority of voters, that candidate will indeed win the election (assuming we aren't using plurality voting, where that is axiomatic)? There are many styles of proofs of theorems about such things (including geometric ones) but most of them have at their heart a contradiction involving allowable profiles and are not overly technical, making them ideal for a wide variety of courses.

Indeed, some other aspects of the field make it quite suitable for many students. For instance, there are many directions to take material, depending on interest. What about elections (say, to a student senate) with multiple winners? How easy is it to manipulate an outcome by voting for a more 'electable' candidate than one's true preference? One can go on, with examples like bill bundling or trying to determine which state truly has the most power in the electoral college; outside of politics, who has the most power in a shareholders' meeting? Further, with slight changes often in the offing (Minneapolis and Vermont being very recent locales to experiment with introducing so-called instant runoff voting), there is a heavy emphasis on synthesizing material with current events. The mathematics of voting and choice is a real application of axiomatic mathematics to an arena not normally associated to it, and studying it can help combat a form of innumeracy – that you should know why your system should be changed, not just because your candidate doesn't win.

A course such as this is worth offering from several standpoints. First, the students may appreciate a number of its characteristics. It is relevant, with major ideas in the news; it requires less mathematical (though not logical) sophistication than a typical math course; it can easily be related to another major, even for credit; and it appeals to students' innate curiosity ('What on earth is the math of voting?!'). Faculty should appreciate the accessibility as well as students, and that examples at the introductory level are easy to construct, with real-time student participation. We should appreciate that in this field real proofs are accessible to non-majors, and that the assumptions behind the theorems and proofs force discussion of underlying ideas and why we make the assumptions we do, a key mathematical idea.

From a more practical point of view, faculty may appreciate that there is a great deal of flexibility in the length and difficulty of material presented. It could be just one topic in a non-major topics course, or used as a high-level topics course with primary documents and papers. It presents itself well to either a discovery-based mode or a more traditional reading-based mode. We combined several strategies in a half-semester course, with a research project into the intersection of the material with the students' own disciplines, and this worked quite well. It is also a good outreach to other departments, as our own experience amply demonstrated, with Political Studies and International Affairs majors receiving course credit – not usually a constituency associated with the math department. A final reason, perhaps more compelling than others for faculty with heavy teaching loads, is that with a variety of good resources (finally) becoming available at all levels, a course in voting theory is possible to offer with a reasonable initial preparation time, even for the non-expert.

In conclusion, a course in voting theory can be a rewarding addition to departmental offerings! It can show math's universal utility on campus; the topic has good flexibility, with enough old results for majors and new enough for non-majors. It is an interesting and compelling field worth exploring at all levels.

We conclude with brief descriptions of all texts with significant content in this area known to the author; others may exist as well.

- *The Heart of Mathematics*, Burger and Starbird
  - This is a popular text introducing students to mathematical ways of thinking, and has a useful but short section on this topic. Useful for supplementing, or for a short section in a longer course for non-majors.
- *For All Practical Purposes*, COMAP Consortium
  - An excellent book on more practical mathematics, with very good chapters on voting theory, game theory, and fair division, all written by undisputed leaders in the field. Worthwhile for a substantial unit in a longer course.
- *The Mathematics of Voting and Elections*, Hodge and Klima
  - The text we used. Suitable for a non-major course (at a low mathematical level) with emphasis in proof or exploration (or both), with entertaining examples and a pseudo-Moore method flavor. About half leading up to Arrow's Theorem, then sections on power indices, weighting, and apportionment.
- *Chaotic Elections*, Saari
  - Explores a wide range of voting issues as well as various approaches, but not a textbook per se. Appropriate for a more reading-focused course, especially if instructor willing to embellish to create exercises, due to wide scope.
- *Fair Representation*, Balinski and Young

    - Standard text in field of apportionment, combining good historical interest with full proofs in appendices. Appropriate for a more research-oriented non-major course, certainly also as a reference for projects.
- *Social Choice and the Mathematics of Manipulation*, Taylor
    - Begins with Arrow, but strongly focused on manipulation, at the cutting edge of research. Fairly steep learning curve even for talented majors, but straightforward at the same time in presentation with few digressions. Good exercises as well.
- *Basic Geometry of Voting*, Saari
    - Foundational research text in geometric approach to the topic, but very accessible to talented undergrads. A real text (slightly dated) with good exercises, and hence very appropriate for a topics course at senior level.