Numerical Range of Toeplitz Matrices over Finite Fields[1]

Derek Thompson, Maddison Guillaume Baker, Amish Mishra (Taylor University)

Derek Thompson (B.S., Indiana Wesleyan University; Ph.D., IUPUI) is an associate professor of mathematics at Taylor University in Upland, IN. He is a Project NExT Gold Dot '14 Fellow, and a member of MAA, ACMS, AMS, and AWM. His research interests are operator theory, linear algebra, and functional analysis. Hobbies include video, board and card games, anime, and medieval fantasy literature. Pictured with Derek (center) are his two student co-authors.

Abstract

This paper characterizes the k-th numerical range of all $n \times n$ Toeplitz matrices with a constant main diagonal and another single, non-zero diagonal, where the matrices are over the field $\mathbb{Z}_p[i]$, with p a prime congruent to $3 \mod 4$. For $k \in \mathbb{Z}_p^*$, the k-th numerical range is always equal to $\mathbb{Z}_p[i]$ with the exception of the scaled identity. Similar techniques are used to discover a general connection between the 0-th numerical range and the k-th numerical range. Lastly, a conjecture is given regarding the general numerical range of all triangular Toeplitz matrices.

1 Introduction

Let p be a prime congruent to $3 \mod 4$. Let $M \in M_n(\mathbb{Z}_p[i])$ where $\mathbb{Z}_p[i]$ is a Galois Field of order p^2 in the form $\{a + bi : a, b \in \mathbb{Z}_p\}$, $M_n(\mathbb{Z}_p[i])$ denotes the set of $n \times n$ matrices with entries from the field in the argument, and M is a Toeplitz matrix. This paper classifies the numerical range of Toeplitz matrices, $W(M)$, over the finite field $\mathbb{Z}_p[i]$. Numerical range of matrices over $\mathbb{C}$ has been of high interest in the mathematical community with substantial advances made in the area by Hausdorff, Toeplitz, and Kippenhahn [2]. However, in this paper we work with the field extension $\mathbb{Z}_p[i]$ of $\mathbb{Z}_p$ where p is prime and $p \equiv 3 \mod 4$, which is based off of a recent publication on numerical ranges over finite fields [2]. Restricting to $p \equiv 3 \mod 4$ ensures that the element -1 is not a quadratic residue in $\mathbb{Z}_p$, reflecting the same property of -1 in $\mathbb{R}$. Note that in this analogue we have preserved the extension field having degree 2.

2 Preliminaries

Before describing the numerical range of Toeplitz matrices in a finite field, we must establish some preliminary definitions and lemmas. We have built the foundation of our research off of a paper by Coons et al ([2]), which contains the specific proofs for this section. We begin by presenting the definition of numerical range over a finite field with p prime and $p \equiv 3 \mod 4$.

Definition 1. [2, Definition 1.1] Let p denote a prime congruent to $3 \mod 4$ and let $M \in M_n(\mathbb{Z}_p[i])$. We define $W(M)$, the **finite field numerical range** of M, to be

$$W(M) = \{x^*Mx : x \in \mathbb{Z}_p[i]^n, x^*x = 1\}.$$

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The reader may note that x^*x is an indefinite inner product since there may be several x that map to 0 under this operation. Therefore, the function x^*x can not be defined as a norm. However, this analogue does maintain several standard properties of the numerical range. The numerical range remains unchanged under scaling and translating, while also remaining unitarily invariant.

Definition 2. [2, Definition 2.2] Let p denote a prime congruent to $3 \mod 4$ and let $||x||^2 := x^*x$ and for any $k \in \mathbb{Z}_p$ let C_n^k denote the set of all vectors $x \in \mathbb{Z}_p[i]^n$ for which $||x||^2 := k$.

The ***k*-th numerical range** of a matrix $M \in M_n(\mathbb{Z}_p[i])$ is the set $W_k(M) = \{x^*Mx : x \in \mathbb{Z}_p[i]^n, x^*x = k\}$.

Lemma 1. [2, Lemma 2.3] *For all primes p congruent to $3 \mod 4$, $k \in \mathbb{Z}_p^*$, and $M \in M_n(\mathbb{Z}_p[i])$, we have $W_k = kW_1(M)$.*

Lemma 2. [2, Lemma 2.7] *Let p be a prime congruent to $3 \mod 4$ and let $M \in M_n(\mathbb{Z}_p[i])$. For any $a, b \in \mathbb{Z}_p[i]$ we have $W(aM + bI) = aW(M) + b$.*

Lemma 3. [2, Lemma 2.6] *Let $M, U \in M_n(\mathbb{Z}_p[i])$ with U unitary and p a prime congruent to $3 \mod 4$. Then, $W(M) = W(U^*MU)$.*

For simplicity, the first numerical range will now be referred to as $W(M)$ as described in Definition 1. It is important to note that Lemma 1 does not apply to $W_0(M)$. Following the work of [2], Ballico classified $W_0(M)$ for every 2×2 matrix in [1]. The work that follows also requires some number-theoretic tools; the following again comes from [2].

Lemma 4. [2, Lemma 2.1] *For all primes p congruent to $3 \mod 4$, and for all $k \in \mathbb{Z}_p$, there exists $t, s \in \mathbb{Z}_p$ for which $t^2 + s^2 = k$.*

There is a nice connection between this theorem and the kind of expressions that constrain the numerical range:

Lemma 5. *Let p be a prime congruent to $3 \mod 4$. For all $k \in \mathbb{Z}_p$ and all $x \in \mathbb{Z}_p[i]$, there exists a $y \in \mathbb{Z}_p[i]$ for which $|x|^2 + |y|^2 \equiv k \mod p$.*

Proof. First, let $x = 0$. Since $0 \in \mathbb{Z}_p[i]$, let $y = 0$. Then, $0^2 + 0^2 \equiv 0 \mod p$, and we are done.

Now, let $x \in \mathbb{Z}_p[i]^*$. It follows that the element $x\overline{x} \in \mathbb{Z}_p^*$. Since the field $\mathbb{Z}_p[i]$ has additive inverses, there exists an $m \in \mathbb{Z}_p^*$ such that $x\overline{x} + m \equiv 0 \mod p$. Therefore, $x\overline{x} + m + k \equiv k \mod p$ when $k \in \mathbb{Z}_p$. It follows that $m + k \in \mathbb{Z}_p$. By Lemma 4, we know there exist $a, b \in \mathbb{Z}_p$ for which $(m + k) = a^2 + b^2$. Letting $y = a + bi$, we have $(m + k) = y\overline{y}$. Therefore, $|x|^2 + |y|^2 \equiv k \mod p$. □

We are now prepared to investigate the numerical range of Toeplitz matrices.

3 Toeplitz Matrices

In this section, we will prove the numerical range of a specific class of Toeplitz matrices. Toeplitz matrices have constant, descending diagonals. The form of a general $n \times n$ Toeplitz matrix M is given below.

$$M = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & \vdots \\ a_2 & a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_1 & a_0 \end{pmatrix}$$

Our Toeplitz matrices have a main diagonal a_0 and a single, non-zero lower or upper diagonal a_r. (If only the main diagonal is non-zero, we already have $W(M) = \mathbb{Z}_p$ by [2, Corollary 3.2].) Here we state our theorem regarding $W(M)$ for such matrices, but its proof will be written as a sequence of several lemmas.

Theorem 3. *Let $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ where M is a Toeplitz matrix with a main diagonal a_0 and a single, non-zero lower or upper diagonal a_r. Then for $k \in \mathbb{Z}_p^*$, $W_k(M) = \mathbb{Z}_p[i]$.*

We now transition into proving the full numerical ranges described in Theorem 3. To begin, we will examine the numerical range of strictly lower triangular, $n \times n$ Toeplitz matrices with a single, non-zero, lower diagonal.

Lemma 6. *For all primes $p \equiv 3 \mod 4$, $W(M) = \mathbb{Z}_p[i]$ where $M \in M_3(\mathbb{Z}_p[i])$ is given by*

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proof. Define $x^* = (c_1\ c_2\ c_3)$, and let $m := x^*Mx = c_1\overline{c_3}$. We will consider a subset of the numerical range by setting $c_3 = 1$. (We will show this subset of $W(M)$ still attains all of $\mathbb{Z}_p[i]$ as outputs.) For all $c_1 \in \mathbb{Z}_p[i]$, there exists an x such that $|c_1|^2 + |x|^2 \equiv 0 \mod p$ by Lemma 5. Let $c_2 = x$. With c_1 varying over all of $\mathbb{Z}_p[i]$ and c_2 always chosen so that $|c_1|^2 + |c_2|^2 \equiv 0$, we have $W(M) = \mathbb{Z}_p[i]$. □

Lemma 7. *For all primes $p \equiv 3 \mod 4$, $W(M) = \mathbb{Z}_p[i]$ where $M \in M_3(\mathbb{Z}_p[i])$ is given by*

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Proof. Consider $m := x^*Mx = c_1\overline{c_2} + c_2\overline{c_3}$. Our goal is to show that m can become any element of $\mathbb{Z}_p[i]$. We will again show this on a subset of the numerical range by stipulating that $c_2 = 1$.

First, we show that there is a non-zero element in this set. Letting $c_1 = 1$, we have that $|c_3|^2 \equiv -1$. By the Lemma 4, there exists $a, b \in \mathbb{Z}_p$ such that $a^2 + b^2 \equiv -1$, so we will let $c_3 = a + bi$. It is important to note that since -1 is not a quadratic residue when $p \equiv 3 \mod 4$, neither a nor b can be 0. Therefore, since c_1 is chosen to be real and c_3 is guaranteed to be complex, we know that $c_1 + \overline{c_3}$ is non-zero, and that the set of all such elements is different from $\{0\}$.

Now, let $c_1 + \overline{c_3}$ be a given fixed non-zero quantity with the constraint that $|c_1|^2 + |c_3|^2 \equiv 0$. Let us now consider inputting the elements kc_1 and $\overline{k}c_3$ where k is an arbitrary element of $\mathbb{Z}_p[i]$. Note that $|kc_1|^2 + |\overline{k}c_3|^2 = |k|^2|c_1|^2 + |k|^2|c_3|^2 = |k|^2(|c_1|^2 + |c_3|^2) = |k|^2(0) = 0$, which satisfies the constraint. Then, the output becomes $kc_1 + k\overline{c_3} = k(c_1 + \overline{c_3})$. Since $c_1 + \overline{c_3}$ is fixed and k varies over all of $\mathbb{Z}_p[i]$, we have that $k(c_1 + \overline{c_3})$ maps to every element of $\mathbb{Z}_p[i]$. This is because $k \to \alpha k$ is an automorphism of $\mathbb{Z}_p[i]$ where $\alpha = c_1 + \overline{c_3} \in \mathbb{Z}_p[i]^*$. Therefore, $W(M) = \mathbb{Z}_p[i]$. □

We now complete the proof to Theorem 3 for the lower diagonal cases by proving the following lemma.

Lemma 8. *For all primes $p \equiv 3 \mod 4$, $W_k(M) = \mathbb{Z}_p[i]$ where $k \in \mathbb{Z}_p^*$ and where $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ is a Toeplitz matrix with a constant diagonal $a_0 \in \mathbb{Z}_p[i]$ and lower diagonal $a_r \in \mathbb{Z}_p[i]^*$ with $0 < r < n$.*

Proof. We first consider $W(M) = \mathbb{Z}_p[i]$ where M is a Toeplitz matrix with a constant diagonal $a_0 = 0$ and lower diagonal $a_r = 1$ with $0 < r < n$.

Suppose $n = 3$. Then the only two cases are already proven to be $\mathbb{Z}_p[i]$ in Lemmas 6 and 7.

Now, suppose $n \geq 4$. Let $x = \begin{pmatrix} c_1 & c_2 & \dots & c_n \end{pmatrix}^T$ where $c_1, c_2, ..., c_n \in \mathbb{Z}_p[i]$ and $|c_1|^2 + |c_2|^2 + ... + |c_n|^2 \equiv 1$. Consider m where $m := x^*Mx = 0\sum_{i=1}^{n} c_i\overline{c_i} + 1\sum_{m=1}^{n-r} c_m\overline{c_{m+r}} = c_1\overline{c_{1+r}} + c_2\overline{c_{2+r}} + ... + c_{n-r}\overline{c_n}$.

We begin by restricting to a subset of the numerical range by requiring $c_{1+r} = 1$. For all $c_1 \in \mathbb{Z}_p[i]$, there exists an x such that $|c_1|^2 + |x|^2 \equiv 0 \mod p$ by Lemma 5. Therefore, let $c_2 = x$ and let all other $c_i \equiv 0$ where $i \neq 1, 2, 1+r$. Therefore, we now have $|c_1|^2+|c_2|^2 \equiv 0$ and $m = c_1(1)+c_2(0) = c_1$. Since we let c_1 vary over all of $\mathbb{Z}_p[i]$, adjusting c_2 appropriately, we have $W(M) = \mathbb{Z}_p[i]$. The only concern is the possibility that $r = 1$, so that $c_{1+r} = c_2$. In that case, since $n \geq 4$, we can let c_3 take on the role of c_2 in the work above, and the proof follows similarly.

By Lemma 2, we know that $W(a_rM + a_0I) = a_rW(M) + a_0 = \mathbb{Z}_p[i]$ since $k \to \alpha k + \beta$ is an automorphism of $\mathbb{Z}_p[i]$. Therefore, $W(M) = \mathbb{Z}_p[i]$ when M is a Toeplitz matrix with a main diagonal $a_0 \in \mathbb{Z}_p[i]$ and a lower diagonal $a_r \in \mathbb{Z}_p[i]^*$ with $0 < r < n$.

Therefore, for the matrix M described, we conclude that $W_k(M) = \mathbb{Z}_p[i]$ where $k \in \mathbb{Z}_p^*$ by Lemma 1. □

We have proven Theorem 3 for all Toeplitz matrices with a single, non-zero lower diagonal. We will now show that the result holds for matrices with an upper diagonal.

Lemma 9. *When $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ is a Toeplitz matrix with a main diagonal a_0 and a single, non-zero, upper diagonal a_r with $0 < r < n$, $W_k(M) = \mathbb{Z}_p[i]$ where $k \in \mathbb{Z}_p^*$.*

Proof. In this setting, we still have $W(M^*) = \overline{W(M)}$. Every Toeplitz matrix M with a single, constant, non-zero upper diagonal can be described by a matrix with a single non-zero lower diagonal by considering M^*. Thus, here again we have $W(M) = \mathbb{Z}_p[i]$. By Lemma 1, one can conclude the same for the k-th numerical range: $W_k(M) = \mathbb{Z}_p[i]$. □

This concludes our proof for Theorem 3.

4 Connecting W_0 to W_k

While W_0 is generally distinct from W_k for $k \neq 0$, our work in this section illustrates an important, general connection between the two, for a much broader class of matrices than simply Toeplitz matrices, inspired by the proofs of the previous section. The main proof depends on direct sums, which work very differently in this setting. The following definition and proposition explain how direct sums now work with numerical range, and are lifted directly from [2].

Definition 4. [2, Definition 2.8] For any two elements s and t of $\mathbb{Z}_p[i]$, we define $L_{s,t}$, the **open line segment** connecting s and t, to be the set $\{sj + t(1-j) : j \in \mathbb{Z}_p, j \neq 0, 1\}$. Furthermore, for any two subsets S and T of $\mathbb{Z}_p[i]$ we define $\operatorname{Conv}(S,T)$, the **open convex hull** of S and T, to be the union of all open line segments connecting an element of S and an element of T, i.e.

$$\operatorname{Conv}(S,T) := \bigcup_{\substack{s\in S\\ t\in T}} L_{s,t}.$$

Finally, we define $\operatorname{Conv}(S,T;S_0,T_0)$, the **oddly-closed convex hull** of S and T with respect to sets S_0 and T_0, to be the union of $\operatorname{Conv}(S,T)$ with sets $S+T_0$ and S_0+T.

Proposition 5. [2, Proposition 3.1] *Let p be a prime congruent to $3 \mod 4$ and let $M \in M_n(\mathbb{Z}_p[i])$. Assume further that M is reducible, i.e. $U^*MU = A \oplus B$ for some unitary matrix $U \in M_n(\mathbb{Z}_p[i])$ and for some lower dimensional matrices A and B with entries in $\mathbb{Z}_p[i]$. Then $W(M)$ is the oddly-closed convex hull of $W(A)$ and $W(B)$ with respect to $W_0(A)$ and $W_0(B)$.*

We are now ready to progress towards our main theorem.

Lemma 10. *Let $A \in M_n(\mathbb{Z}_p[i])$ and let B be the block matrix $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. Then $W(B) = \bigcup_{k\in\mathbb{Z}_p} W_k(A)$.*

Proof. Here $S = W(A), T = W(0) = \{0\}, S_0 = W_0(A)$, and $T_0 = W_0(0) = \{0\}$. Therefore $S + T_0 = W(A)$ and $T + S_0 = W_0(A)$. Additionally, we have $\operatorname{Conv}(S,T) = \bigcup W_k(A)$ where $k \in \mathbb{Z}_p \neq 0, 1$. Taking the union of all these sets as specified in Proposition 5 gives the result. ☐

Lemma 11. *Let $A \in M_n(\mathbb{Z}_p[i]), n \geq 2$, with at least one non-zero entry a_{jk} which is not on the main diagonal, and $a_{kj} = a_{jj} = a_{kk} = 0$. Then the block matrix $B = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ has $W(B) = \mathbb{Z}_p[i]$.*

Proof. Consider a subset of the numerical range by requiring inputs x such that $c_k = 1$, and $c_i = 0$a for $i \neq j, k, n+1$. Then, the expression x^*Ax becomes $a_{jk}c_j\overline{c_k} + a_{kj}c_k\overline{c_j} = a_{jk}c_j$, with the constraint that $|c_j|^2 + |c_{n+1}|^2 = 0$. Letting c_j be any element of $\mathbb{Z}_p[i]$, and adjusting c_{n+1} appropriately as in the proof of Lemma 6, we have $W(B) = \mathbb{Z}_p[i]$. ☐

Theorem 6. *Let $A \in M_n(\mathbb{Z}_p[i]), n \geq 2$, with at least one non-zero entry a_{jk} which is not on the main diagonal, and $a_{kj} = a_{jj} = a_{kk} = 0$. Then*

$$\bigcup_{k\in\mathbb{Z}_p} W_k(A) = \mathbb{Z}_p[i].$$

Proof. The theorem is a direct consequence of the previous two lemmas. □

This theorem shows that for a large class of matrices, if the numerical range is not all of $\mathbb{Z}_p[i]$, then the missing elements will be found in *some* variation of the numerical range. Our constraint on the matrix - a non-symmetric 0 somewhere - may seem odd, and it is not clear how sharp this constraint is. However, the theorem certainly fails for symmetric matrices. For example, computing x^*Mx where $M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $x = (c_1 c_2 c_3)$ gives $c_1\overline{c_2} + \overline{c_1}c_2 + c_2\overline{c_3} + c_3\overline{c_2}$ which is clearly an element of $\mathbb{Z}_p$, regardless of the constraint on $\|x\|^2$.

The theorem also gives an immediate corollary for when W_0 might "compensate" for missing elements of $W(M)$.

Corollary 7. *Let $A \in M_n(\mathbb{Z}_p[i]), n \geq 2$, with at least one non-zero entry a_{jk} which is not on the main diagonal, and $a_{kj} = a_{jj} = a_{kk} = 0$. Suppose further that $W(A)$ is either $\mathbb{Z}_p$ or $\{x + xi : x \in \mathbb{Z}_p\}$. Then $W_0(A)$ contains the complement of $W(A)$, and $W_0(A) \cup W(A) = \mathbb{Z}_p[i]$.*

Proof. In either of these situations, $W_k(A) = W(A)$ for all $k \in \mathbb{Z}_p^*$. □

5 Future Work

Throughout our research, we have found that the standard numerical range of lower or upper triangular Toeplitz matrices is always full. For some matrices, this is easily proven with the techniques we have used thus far. For example, consider

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

and $x = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{pmatrix}^T$ where $c_1, c_2, c_3, c_4, c_5 \in \mathbb{Z}_p[i]$ and $|c_1|^2+|c_2|^2+|c_3|^2+|c_4|^2+|c_5|^2 = 1$. The elements of the numerical range are in the form $m = x^*Mx = c_1\overline{c_2} + c_2\overline{c_3} + c_1\overline{c_4} + c_3\overline{c_4} + c_1\overline{c_5} + c_2\overline{c_5} + c_4\overline{c_5}$. If we restrict x such that c_1 and $c_5, = 0$ and $c_3 = 1$, we are left with $m = c_2(1) + (0)\overline{c_4}$ when $|c_3|^2 + |c_4|^2 = 0$. This was proven to be $\mathbb{Z}_p[i]$ in Lemma 6.

It seems as if we can classify every upper or lower triangular Toeplitz matrix with a full numerical range if there is at least one off-diagonal of zeroes. The problem arises from a matrix where we have no such diagonal of zeroes. For example, consider the following matrix

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

An arbitrary element of $W(M)$ is in the form $m = c_1\overline{c_2} + c_2\overline{c_3} + c_1\overline{c_3}$. There is no simple trick we know of to verify this element gives us all of $\mathbb{Z}_p[i]$. But our testing has shown that for small $p \equiv 3 \mod 4$, we know this does give us all of $\mathbb{Z}_p[i]$.

From this example, and several like it, we make the following conjecture.

Conjecture. *For all primes $p \equiv 3 \mod 4$, $W(M) = \mathbb{Z}_p[i]$ where $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ is either a lower or upper triangular Toeplitz matrix.*

Additionally, as mentioned earlier, W_0 is distinct from W_k. However, in all of our work, we have found $W_0(M) = W(M)$ when $W(M) = \mathbb{Z}_p[i]$. A proof of the following conjecture would also be of interest.

Conjecture. *For all primes $p \equiv 3 \mod 4$, $W_0(M) = \mathbb{Z}_p[i]$ where $M \in M_n(\mathbb{Z}_p[i]), n \geq 3$ is either a lower or upper triangular Toeplitz matrix.*

We have also tested non-triangular matrices and have seen, but are unable to show, that in many cases (but not all, e.g. if the matrix is symmetric), $W(M) = \mathbb{Z}_p[i]$. In general, for higher dimensional matrices, it seems that finding a numerical range other than $\mathbb{Z}_p[i]$ is rare, even when a Toeplitz form is not assumed.

We conclude with a list of further questions.

1. Is it true that every triangular Toeplitz matrix which is not a multiple of the identity has $W(M) = \mathbb{Z}_p[i]$?
2. If a Toeplitz matrix is symmetric, then $W(M)$ is a line. Is it true that for every Toeplitz matrix, $W(M)$ is either a line or $\mathbb{Z}_p[i]$? If not, what other regions are possible?
3. If T is a multiple of the identity, then $W(M) \neq W_0(M)$. Is this ever true otherwise for Toeplitz matrices?
4. How different is the situation when $p \not\equiv 3 \mod 4$?

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