# Bach (to the Calculus of) Variations

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First, a few comments about the title: Originally I used it as a hook to get students interested in a colloquium talk. I wanted them to wonder what Bach variations had to do with calculus or more generally how music and math might be related. I believe the title worked based on the attendance at undergraduate colloquia at several institutions.

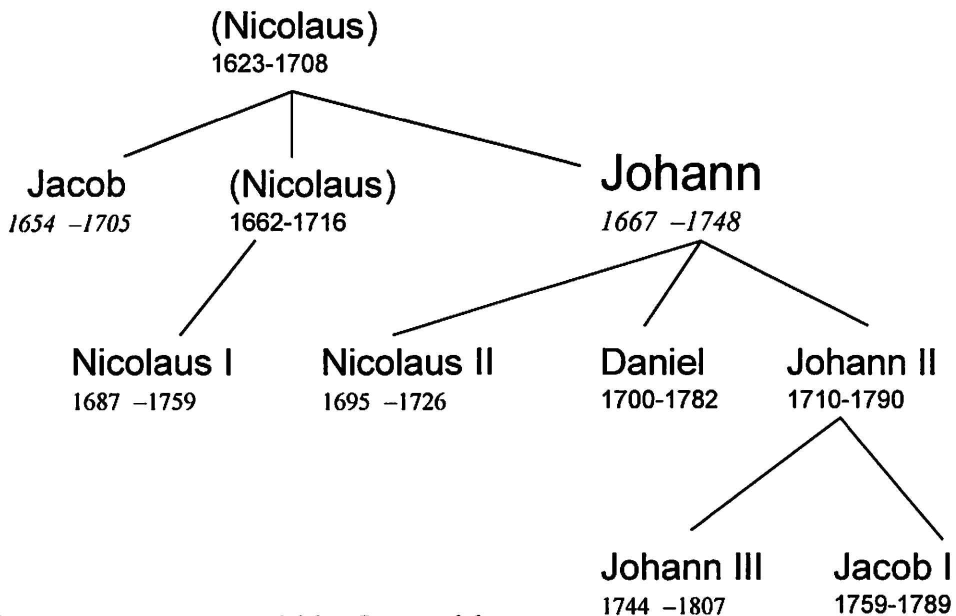
While it is quite common for professionals (doctors, lawyers, academics, etc) to be talented in many ways, including musical talent, there is a special connection between music and mathematics. Musicians collectively are not more talented in mathematics than other professionals, but mathematicians do seem to have more musical talent than other professionals and other academics. This group, the ACMS, is a prime example. Our shared Christian commitment may skew the sample, but I maintain that even a random MAA sectional meeting would have more musical talent among its number than a regional group of historians, or chemists, or dentists. Right here in this room we have several people with considerable musical talent/gifts and even a few who have studied music seriously enough that the decision to be a mathematician rather than a musician was not an easy one.

There are undoubtedly many who remember the Presidents' Concert at the 1989 AMS-MAA Joint Meeting in Phoenix given by the Leonard Gillman, pianist and President of the MAA, and William Browder, flutist and President of the AMS. These same individuals reprised the concert in a Past-Presidents' Concert at the 1992 AMS-MAA joint Meeting in Baltimore. Also a few readers, who did graduate study at the University of Wisconsin, will recall the biannual brown-bag noon-time piano recitals given by Larry Levy, Professor of Mathematics. Professor Levy completed an MA in piano performance at Julliard before switching careers to mathematics.

This math-music connection makes it easier to understand that there might be a connection between Johann Sebastian Bach and the mathematician most closely associated with the beginning of the subject of the Calculus of Variations, Johann Bernoulli.

1. They are contemporaries: **Johann Sebastian Bach (1685-1750)**
   **Johann Bernoulli (1667-1748)**
2. Each lived and worked in German speaking Europe:
   **Bach in various parts of northern Germany**
   **Bernoulli in Basel, Switzerland**
3. Each was the most prominent member of an extended family with many members distinguished in their respective fields.

The Bernoulli family has its origins in Belgium/Holland and were in the spice import and distribution business. They moved to Basel, Switzerland when King Phillip of Spain made life for Protestants - the Bernoullis were Calvinists - very unpleasant in Holland. Brothers Jacob and Johann were the first to show significant mathematical talent.




The Bernoulli family

Music was the Bach family business. Both prior to J. S. Bach and subsequent to him family members, both men and women, were church musicians, composers, and/or court musicians.

Johann Ambrosius

Johann Sebastian
1685-1750

(Maria)

Wilhelm Friedemann
1710-1784

Carl Philipp Emannuel
1714-1788

(Anna)

Gottfried Heinrich
1724-1763

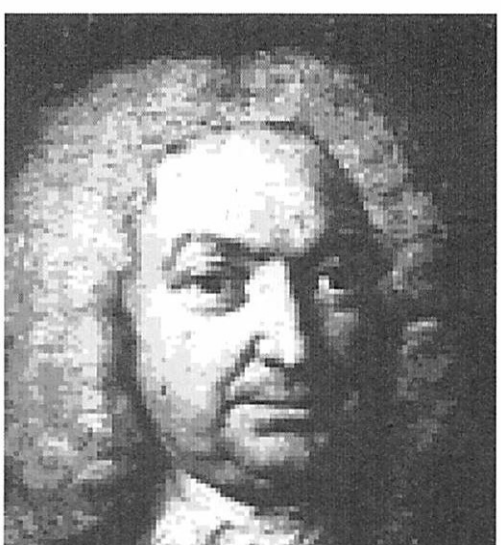
Johann Christian
1735-1782

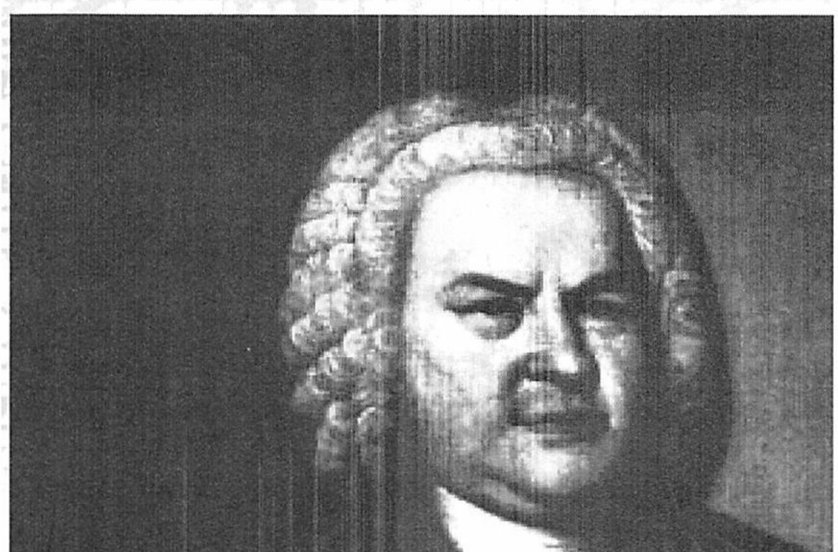
Johann Christoph Friedrich
1732-1795

Wilhelm Friedrich Ernst
1759-1845

# The Bach family

4. J. S. Bach and Johann Bernoulli even look like brothers. Can you tell who is who?





The similarity above is largely due to 18th Century formal dress and the practice of portraits being done in a quarter face on manner. Bach is on the right.

In a previous version of this paper I raised the questions of 1) whether these two Johanns had ever met each other or if 2) members of their families had done so. My answers were: 1) probably not and 2) quite likely so. Further research has led me to the following more definitive answers.

1. Although J. S. Bach lived in several places, he never left northern Germany. Johann Bernoulli grew up in Basel, lived for a time in Paris where he taught calculus to L'Hopital and sold to him the famous theorem. He also worked at the university in Groningen, Holland before returning to Basel in 1705 to take up the chair in mathematics upon the death of his brother Jacob. Thus I am convinced the two Johanns never met but, given the small size of the intellectual class in 18th century Europe, quite likely knew of each other.

2. An answer to the second question comes from looking at the court of Frederick the Great in Potsdam/Berlin. J. S. Bach's son, Carl Philipp Emmanuel Bach was a musician, playing the clavier/harpsicord in the court of the King Frederick from 1740 – 1768. Johann Bernoulli III, the grandson of Johann, was a member of Frederick's Berlin Academy of Science from 1763 – 1807. The young Johann Bernoulli was almost surely acquainted with the much - 30 yrs - older C. P. E. Bach. If we include in the Bernoulli family Johann Bernoulli's mathematical "son", Leonhard Euler, then we certainly have a connection. Euler spent the years 1741 – 1766 at the Berlin Academy. Although I have seen no direct evidence of a social relationship during those 25 years, Euler and C. P. E. Bach certainly knew each other and must have often been at court functions together. I will leave it to cultural historians to uncover the parameters of the relationship. I have discovered that J. S. Bach himself visited his son at the court in Potsdam twice, once in 1741 and again in 1749 and might well have met Euler on those visits.

This cultural history, leads us to the Calculus of Variations. In the first third of the 20th century every serious undergraduate math major would have been introduced to the Calculus of Variations and it was a significant topic of graduate school programs and research, e. g. at Chicago in the 1920's. However, it is almost never appears in the undergraduate curriculum today and has been subsumed within the area of Optimal Control Theory.

It is a shame it has disappeared, since it is a beautiful part of mathematics, it is a relatively easy application of calculus, and is connected to historically important problems. One of these problems, the brachystochrone, is mentioned in virtually every calculus text but never solved in them. In fact the 2004 publication [Van Brunt] by Springer-Verlag is the first introductory text on the topic in at least twenty years.

So what is the Calculus of Variations? Let's begin by looking at three problems.

**A.** Find the curve of shortest distance between two points in the plane.

Let the two points be (0,0) and $(x_1,y_1)$. Choose a function y = f(x) so that the f(0) = 0, $f(x_1) = y_1$, and which minimizes the arc length integral

$$I = \int_0^{x_1} \sqrt{1+(y')^2}\, dx.$$

**B.** Find the curve between two points which produces the minimum surface of revolution.

Let the axis of rotation be the x-axis and the two points be (0,0) and $(x_1,y_1)$. Choose a function y = f(x) so that the f(0) = 0, $f(x_1) = y_1$, and which minimizes the surface area integral

$$I = \int_0^{x_1} 2\pi y\, ds = 2\pi \int_0^{x_1} y\sqrt{1+(y')^2}\, dx.$$

**C.** Find the curve between two points along which the descent is quickest.
( the brachystochrone or brachistochrone problem)

Let the two points be (0,0) and $(x_1,y_1)$ and for convenience let the positive y-axis be oriented downward. For simplicity assume zero initial velocity and ignore all forces except gravity, i.e. the motion is frictionless. From conservation of energy we have

$$\frac{1}{2}mv^2 = mgy \quad OR \quad v = \sqrt{2gy}$$

Choose a function y = f(x) so that the f(0) = 0, $f(x_1) = y_1$, and which minimizes

$$I = \int_0^{x_1} dt = \int_0^{x_1} \frac{ds}{v} = \int_0^{x_1} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\, dx$$

These three problems all involve minimizing an integral of the following form

$$I = \int_{x_o}^{x_1} \phi(x,y,y')dx$$

Calculus of Variations provides us with a general approach to minimizing integrals of this type. Suppose y = f(x) is the minimizing function and y = F(x) is another function fitting the boundary conditions. Let $\eta(x) = F(x) - f(x)$ be the perturbation function. Then for $y = f(x) + a\,\eta(x)$ we have y = f(x) when a = 0 and y = F(x) when a = 1. Also note that the integral is a function only of the parameter a.

$$I(a) = \int_{x_o}^{x_1} \phi\big(x, f(x)+a\eta(x), f'(x)+a\eta'(x)\big)dx$$

If y = f(x) is the minimizing function, then this integral musty have a minimum value at a = 0 for any function $\eta(x)$ satisfying the boundary conditions $\eta(x_0) = 0 = \eta(x_1)$.

As good calculus students we minimize $I(a)$ by setting $I'(a) = 0$. If the integrand is nicely behaved, uniformly continuous with partial derivatives uniformly continuous for instance, then

$$I'(a) = \frac{d}{da}\int_{x_o}^{x_1} \phi(x, f(x) + a\eta(x), f'(x) + a\eta'(x))dx$$

$$= \int_{x_o}^{x_1} \frac{d}{da}\left[\phi(x, f(x) + a\eta(x), f'(x) + a\eta'(x))\right]dx$$

$$I'(a) = \int_{x_o}^{x_1} \frac{\partial\phi}{\partial y}\eta + \frac{\partial\phi}{\partial y'}\eta' dx$$

Integrating the second term above by parts gives us

$$I'(a) = \int_{x_o}^{x_1} \frac{\partial\phi}{\partial y}\eta\, dx + \left(\frac{\partial\phi}{\partial y'}\eta\right)_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{\partial\phi}{\partial y'}\right)\eta\, dx$$

The boundary conditions eliminate the middle term above to yield

$$I'(a) = \int_{x_o}^{x_1}\left(\frac{\partial\phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial\phi}{\partial y'}\right)\right)\eta(x)\, dx$$

Since we are assuming that a minimum exists at a = 0, $I'(0) = 0$ independent of what the perturbation function $\eta(x)$ happens to be, we are forced to conclude that on

the interval $(x_0, x_1)$

$$\frac{\partial\phi}{\partial y} - \frac{d}{dx}\left(\frac{\partial\phi}{\partial y'}\right) = 0$$

This partial differential equation is called the **Euler-Lagrange** equation and must be satisfied by the integrand $\phi(x,y,y')$ and our minimizing function $y = f(x)$.

Let's use this Euler-Lagrange equation to solve our three problems.

**A. Shortest distance:** Minimize $I \;=\; \int_0^{x_1} \sqrt{1+(y')^2}\,dx$

In this case the Euler-Lagrange equation becomes $\dfrac{d}{dx}\left(\dfrac{y'}{\sqrt{1+(y')^2}}\right)=0$

From which we see that $\dfrac{y'}{\sqrt{1+(y')^2}} \;=\; C \;\Rightarrow\; y' \;=\; K \;\Rightarrow\; y \;=\; Kx+L.$

This is the equation of a straight line as we expected. Thus if a minimal distance curve exists it must be a straight line.

To prove that this is a minimum, consider two curves $\bar{y}=Kx+L$ and a perturbation $\bar{y}+\eta$. The difference between the lengths of these two curves is

$$\int_o^{x_1} \sqrt{1+\left(\bar{y}'+n'\right)^2}\,dx \;-\; \int_o^{x_1} \sqrt{1+\left(\bar{y}'\right)^2}\,dx$$

Expanding the first term above using Taylor's Theorem the above difference becomes for some $0<\alpha<1$

$$= \int_o^{x_1} \frac{\bar{y}'}{\sqrt{1+(\bar{y}')^2}}\eta' + \frac{1}{2}\left(\frac{1}{\sqrt{1+(\bar{y}'+\alpha\eta')^2}}\right)^3 \eta'^2 dx$$

$$= C\eta\Big|_{x_0}^{x_1} \;+\; \int_{x_0}^{x_1} \frac{1}{2}\frac{\eta'^2}{\left(1+(\bar{y}'+\alpha\eta')^2\right)^3}dx$$

The first term above is zero and the second is an integral of a positive quantity unless $\eta' \equiv 0$, which by the boundary conditions would mean that $\eta \equiv 0$.

That is, unless $\eta \equiv 0$ we can conclude that the second curve has longer length.

**B. Minimal surface of revolution:** Minimize

$$I \;=\; \int_0^{x_1} 2\pi y\,ds \;=\; 2\pi \int_0^{x_1} y\sqrt{1+(y')^2}\,dx$$

Here the Euler-Lagrange equation becomes

$$\sqrt{1+(y')^2} \;-\; \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+(y')^2}}\right) \;=\; 0$$

**OR**

$$0 = \sqrt{1+(y')^2} - \frac{(y')^2}{\sqrt{1+(y')^2}}$$

$$- \frac{yy''}{\sqrt{1+(y')^2}} - \frac{y(y')^2 y''}{\left(1+(y')^2\right)^{3/2}}$$

This looks positively awful ! But if we simplify by removing the denominators, we have

$$0 = (1+(y')^2)^2 - (y')^2(1+(y')^2) - yy''(1+(y')^2) - y(y')^2 y''$$

This further simplifies to $0 = 1 + (y')^2 - yy''$ which is a relatively easy 2nd order ordinary differential equation. Since the independent variable, x, is missing the following substitution

$$w = y' = \frac{dy}{dx} = f(y), \qquad y'' = \frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx} = w'w$$

gives us a separable 1st order equation. $1 + w^2 = yw'w = y\dfrac{dw}{dy}w$

This is easily solved yielding $Ky = \sqrt{1+w^2}$.

Now solving for $w$ we get another separable 1st order equation.

$$y' = w = \sqrt{\frac{y^2}{a^2} - 1} = \frac{1}{a}\sqrt{y^2 - a^2}$$

This time the integration is not so easy to do.

$$\int \frac{dy}{\sqrt{y^2 - a^2}} = \int \frac{dx}{a}$$

While trig substitution will yield an equivalent solution involving the natural logarithm, using MAPLE to integrate we see that

$$\cosh^{-1}\left(\frac{y}{a}\right) = \frac{x}{a} + b \quad \text{or equivalently} \quad y = a\cosh\left(\frac{x}{a} + b\right).$$

You ought to recognize this curve as a catenary, the shape taken by a wire hanging between two telephone poles. Thus the curve with minimum surface area is a catenary.

**C. Curve of quickest descent:** Minimize $I = \int_0^{x_1} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\,dx$

Here the Euler-Lagrange equation becomes

$$-\frac{1}{2}\frac{\sqrt{1+(y')^2}}{y^{3/2}} - \frac{d}{dx}\left(\frac{y'}{y^{1/2}\sqrt{1+(y')^2}}\right) = 0$$

**OR**

$$\frac{1}{2}\frac{\sqrt{1+(y')^2}}{y^{3/2}} + \frac{y''}{y^{1/2}\sqrt{1+(y')^2}}$$

$$-\frac{1}{2}\frac{(y')^2}{y^{3/2}\sqrt{1+(y')^2}} - \frac{(y')^2 y''}{y^{1/2}(1+(y')^2)^{3/2}} = 0$$

This is really awful. But simplifying as before leads to an equation very much like the previous problem. $1+(y')^2+2yy''=0$ Using the same substitution used in the minimal surface problem our solution proceeds as follows.

$$1+w^2+2yw'w=0$$

$$K = y(1+w^2) = y(1+(y')^2)$$

$$y' = \frac{\sqrt{K-y}}{\sqrt{y}}$$

$$\int \frac{\sqrt{y}dy}{\sqrt{K-y}} = \int dx \quad \text{OR} \quad \int \frac{ydy}{\sqrt{Ky-y^2}} = x$$

This last integral can be done by hand or by using MAPLE but both lead to very messy expressions involving both $\sqrt{Ky-y^2}$ and $\tan^{-1}\left(\sqrt{Ky-y^2}\right)$ and so is not very insightful. Below is anther way to do the integral which is quite informative.

For $0 \le \theta \le 2\pi$ let $y = K\sin^2\frac{\theta}{2}$. Then $\int \frac{ydy}{\sqrt{Ky-y^2}} = \int \frac{\sqrt{y}}{\sqrt{K-y}}dy$

$$= \int \frac{\sqrt{K}\sin\frac{\theta}{2}}{\sqrt{K}\cos\frac{\theta}{2}} K\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta = K\int \sin^2\frac{\theta}{2}d\theta = K\int \frac{1-\cos\theta}{2}d\theta = \frac{K}{2}(\theta-\sin\theta)$$

Thus in terms of the parameter $\theta$ our solution for the brachystochrone problem is

$$x = \frac{K}{2}(\theta - \sin\theta) + C$$

$$y = K\sin^2\frac{\theta}{2} = \frac{K}{2}(1-\cos\theta)$$

We find that C = 0 by using the initial condition that when $\theta = 0$ , (x,y) = (0,0). This is the familiar set of parametric equations for a cycloid made from a circle of radius $K/2$ .

In the last two problems the reader is left with the task of showing that a catenary curve and a cycloidal curve can be found that satisfy the initial conditions of the respective problems. Analysis can also show that these minimal curves are unique.

I hope that this brief excursion into the Calculus of Variations has convinced you that the subject should not be forgotten. It seems to me that at a minimum this topic is a good project for a senior math major.

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