The *Best* Religious Calendar

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Abstract

We show that the phases of the moon loosely cycle with short period 19 years and long period 160 years. Counterintuitively, the variation in the short period is over five times larger than the variation in the long period. We give a rationale as to why this ratio is so large involving what we call the *signature* of a real number and the *continued fraction* of a real number.

Hooray! It's Hanukkah![1] It's Easter![2] It's Ramadan![3] It's Diwali![4] It's Durin's Day![5] It's Tét![6] When next will such a day arrive on the same date with respect to the seasons, and, in particular, the Gregorian calendar?

The *Quest* or answer to this question will be that integer p years for which the standard deviation between new moons p years apart is minimal, provided p is not too large. Does such an integer exist? Isn't the moon's cycle chaotic?

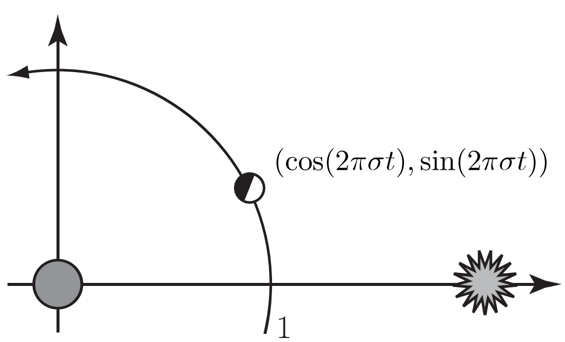
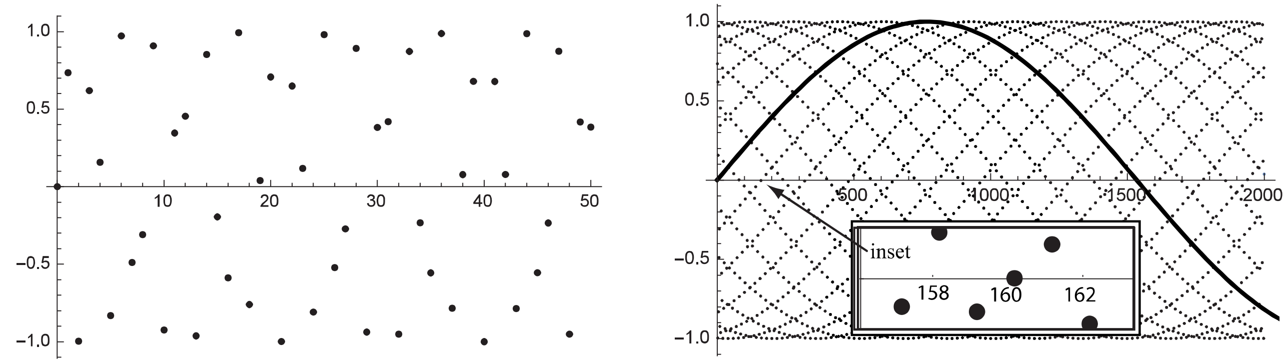


Figure 1: The earth-moon-sun model

A Simple Earth-moon-sun Model

The question of the frequency at which the moon cycles is rooted in the earliest days of mankind. One difficulty with characterizing the moon's motion is that it involves the three body problem. After Isaac Newton derived Kepler's laws from first principles assuming an inverse square law of gravitation, he focused on the earth, moon, and sun, so as to determine where the moon would be at any time and ultimately gave up, saying to Edmund Halley that

[1] A Jewish holiday starting on the 25^{th} day of the lunar month Kislev, where the beginning of each month is a new moon.
[2] A Christian holiday on the first Sunday after the full moon following the spring equinox.
[3] A month long Muslim fasting holiday starting with the new moon that initiates the lunar month of Ramadan.
[4] A Hindu holiday whose zenith is the new moon between mid-October and mid-November.
[5] A fictional realm of Middle Earth holiday starting on the first day of the last moon of autumn [7].
[6] Also known as the Chinese New Year, which is usually the second new moon after the winter solstice.



(a) A smattering of dots

(b) Connecting a strand of dots

Figure 2: The moon's signature, $\mathcal{S}_\sigma$

> The three body problem had "made his head ache, and kept him awake so often, that he would think of it no more." (Steven Strogatz [6, p. 160])

Although the mean time it takes the moon to complete one circuit of the earth with respect to the sun is about 29.53 days (the *synodic* period), the exact time varies up to about 7 hours from this mean. Indeed, in 1887, Henri Poincaré showed the futility of searching for an analytic lunar cycle formula, that the very pattern is one of chaos. Of course, we can dynamically number crunch and extend our predictions of the earth's and the moon's position to a reasonable degree of tolerance arbitrarily far into the future as NASA has done, logging the dates of the four quarters of the moon over a 6000 year period, [2].

In spite of Poincaré's observation—which might doom the very foundation of our Quest—we model the motion of the earth, moon, and sun, using the simplest non-trivial model imaginable and hope that analyzing such a model leads us nearer our Quest. Today, from NASA websites, the moon's mean period around the earth is $T_m \approx 27$ days, 7 hours, 43 minutes, and 11.5 seconds $= 2\,360\,591.5$ seconds (the *sidereal* period) and the earth's period around the sun with respect to the relatively fixed starry background is $T_s \approx 365$ days, 6 hours, 9 minutes, and 9.5 seconds $= 31\,558\,149.5$ seconds. Even though we have long since abandoned a geocentric model of the solar system—to keep the corresponding dynamics simple—we imagine the sun as going around the earth with period T_s. Thus the relative angular velocity σ of the sun to the moon is $\sigma = T_s/T_m \approx 13.368747$.

Again, to keep the dynamics simple, we fix the sun as well as the earth in place so that the relative angular velocity of the moon around the earth is $\sigma_0 = \sigma - 1 \approx 12.368747$, a value agreeing with our experience of having twelve moons (months) per year. With respect to the earth and the sun under these assumptions, the moon's position with respect to time t is *simple harmonic* motion given by

$$(\cos(2\pi\sigma t),\, \sin(2\pi\sigma t)) \tag{1}$$

wherein the moon is unit distance from earth's center. To analyze when new moons reprise in our system, it is sufficient to study only the second component of (1), $w(t) = \sin(2\pi\sigma t))$, because a new moon occurs at each of w's roots when the graph of w crosses the t-axis from negative to positive. For any positive real number α, we say that α's *signature* $\mathcal{S}_\alpha$ is

$$\mathcal{S}_\alpha = \{(n,\, \sin(2\pi\alpha n))|n \in \mathcal{N}\},$$

where $\mathcal{N}$ is the set of integers, first defined in [4] and [5]. In terms of our model, $\mathcal{S}_\sigma$ logs how far the moon is from being new or full at precisely n years later than the initial time, $t = 0$ years. When we peek at $\mathcal{S}_\sigma$ over a short period of years, such as 50 years, we see no real pattern in the arrangements of the signature's dots, as shown in Figure 2a, but when we take a long view over, say, 2000 years, the dots seem to sort themselves into 19 sine-like strands, with successive strands being 160 year translates of each other, as shown in Figure 2b, in which we have connected adjacent dots along the strand containing the origin. That is, if time z years is the date of a new moon, then $z \pm 19$ years are also close to being new moon dates because $w(z) \approx 0$ and $(z \pm 19,\ w(z \pm 19))$ are adjacent strand points with $(z,\ w(z))$, which means that $w(z \pm 19) \approx 0$. Furthermore, $(z + 160,\ w(z + 160))$ is on the strand adjacent to $(z,\ w(z))$'s strand and $w(z + 160) \approx 0$, which means that at time $t = z + 160$, the moon should almost be new.

That is, our Quest answer has dwindled to two options, $p = 19$ years or $p = 160$ years. Which is the better value in our model?

Checking the Simple Earth-moon-sun Model Against NASA Data

The observation that the moon's phases cycle with period 19 years dates at least to the fifth century BC Athens astronomer Meton[7] who championed a new calendar based upon a 19 year cycle. Indeed the Babylonians probably knew about this 19 year cycle as well. And all of the world's lunar holidays, including Easter, are on a 19 year cycle. Did they/we get it correct?

To determine this toss-up, we contrast the 19 year cycle and 160 cycle using data available from NASA's website [2]. We say that a *short span* is an ordered pair of new moon dates approximately 19 years apart, and a *long span* is an ordered pair of new moon dates approximately 160 years apart with the first components being less than the second components.

Table 1 is a listing of 30 non-overlapping short spans ranging from 1480 to 2099 wherein each span contains precisely four leap years. The asterix marking the year 1600 in the table's first column serves to alert the reader of the 1582 calendar change from the *Julian Calendar*—in which every fourth year is a leap year—to the *Gregorian Calendar*—in which every fourth year is a leap year *except* at century years non-divisible by 400, called *deficient* centuries. That is, year 1700 is a deficient century, but not year 2000. The central column of this table gives the time difference in hours, *modulo* 19 years, between the short span dates: second component minus first component. Thus, for example in the first row of the table, the difference between 3 November 1499 and 2 November 1480 is 24.63 hours. The mean and standard deviation of these 30 *short span differences* are $\bar{x}_1 \approx 16.16$ and $s_1 \approx 7.79$ hours, where each short span contains exactly four leap years.

Table 2 is a listing of 30 non-overlapping long spans ranging from year 2000 BC through year 2881 AD. In the first column some years have asterisks attached, such as 1601* and 1761**. The notation n* means that the long span $(n,\ n + 160)$ contains exactly one deficient century. For example, between 1601 and 1761, the year 1700 is deficient. The notation n** means that the corresponding long span contains exactly two deficient centuries. For example, between 1761 and 1921, both 1800 and 1900 are deficient. Thus, the time differential within a single-asterisk-long span must be reduced by 24 hours, and within a double-asterisk-long span by 48 hours. The central column of this table reflects this adjustment. Note also that no long span in the table contains the year 1582, the year the Gregorian Calendar supplanted the Julian Calendar in our time reckoning.

The mean and standard deviation of the *long span differences* of these 30 long spans are $\bar{x}_2 \approx 25.62$ hours and $s_2 \approx 3.31$ hours. The ratio of s_1 to s_2 is about 2.35; equivalently, the ratio of their variations is over 5.5. In trials of 30 alternate short and long spans, this ratio waxed higher at times. Does our simple model anticipate a variation ratio this high, or is this ratio inflated because of chaos?

[7] Meton is immortalized in Aristophanes' play *The Birds* in which he makes a brief appearance as a comic circle-squarer accoutred with compasses and straight-edges [1, p. 304].

Table 1: NASA's New-moon Dates 19 Years Apart

α	hour lapse	$\alpha + 19$ years
2 Nov 1480 @ 8:48	24.633	3 Nov 1499 @ 9:26
22 Oct 1500 @ 23:23	15.683	23 Oct 1519 @ 15:04
11 Oct 1520 @ 15:54	8.117	12 Oct 1539 @ 0:01
29 Oct 1540 @ 20:50	18.667	30 Oct 1559 @ 15:30
19 Oct 1560 @ 6:49	24.000	20 Oct 1579 @ 6:49
5 Nov 1600* @ 22:53	8.150	6 Nov 1619 @ 7:02
25 Oct 1620 @ 13:55	21.983	26 Oct 1639 @ 11:54
15 Oct 1640 @ 4:10	22.183	16 Oct 1659 @ 2:21
3 Nov 1660 @ 1:06	11.300	3 Nov 1679 @ 12:24
22 Oct 1680 @ 11:55	11.317	22 Oct 1699 @ 23:14
12 Oct 1700 @ 10:16	24.017	13 Oct 1719 @ 10:17
31 Oct 1720 @ 11:42	21.883	1 Nov 1739 @ 9:35
20 Oct 1740 @ 16:35	8.150	21 Oct 1759 @ 0:44
9 Oct 1760 @ 1:36	−8.600	8 Oct 1779 @ 17:00
27 Oct 1780 @ 17:10	24.167	28 Oct 1799 @ 17:20
18 Oct 1800 @ 8:58	18.717	19 Oct 1819 @ 3:41
6 Nov 1820 @ 0:08	8.067	6 Nov 1839 @ 8:12
25 Oct 1840 @ 8:59	15.567	26 Oct 1859 @ 0:33
14 Oct 1860 @ 14:38	24.517	15 Oct 1879 @ 15:09
2 Nov 1880 @ 15:55	18.533	3 Nov 1899 @ 10:27
23 Oct 1900 @ 13:27	7.200	23 Oct 1919 @ 20:39
12 Oct 1920 @ 0:50	19.667	12 Oct 1939 @ 20:30
30 Oct 1940 @ 22:03	24.633	31 Oct 1959 @ 22:41
20 Oct 1960 @ 12:02	14.350	21 Oct 1979 @ 2:23
9 Oct 1980 @ 2:50	8.733	9 Oct 1999 @ 11:34
27 Oct 2000 @ 7:58	19.667	28 Oct 2019 @ 3:38
16 Oct 2020 @ 19:31	23.633	17 Oct 2039 @ 19:09
4 Nov 2040 @ 18:56	14.250	5 Nov 2059 @ 9:11
24 Oct 2060 @ 9:25	8.900	24 Oct 2079 @ 18:19
13 Oct 2080 @ 2:44	22.800	14 Oct 2099 @ 1:32
average lapse:	16.160	st.dev: 7.79 hours

Table 2: NASA's New-moon Dates 160 Years Apart

α	hourly lapse	$\alpha + 160$ years
5 Nov 2000 BC @17:15	26.117	6 Nov 1840 BC @19:22
8 Oct 1840 BC @6:34	29.717	9 Oct 1680 BC @12:17
9 Oct 1680 BC @12:17	30.067	10 Oct 1520 BC @18:21
10 Oct 1520 BC @18:21	29.333	11 Oct 1360 BC @23:41
11 Oct 1360 BC @23:41	28.017	13 Oct 1200 BC @3:42
13 Oct 1200 BC @3:42	26.300	14 Oct 1040 BC @6:00
14 Oct 1040 BC @6:00	23.900	15 Oct 880 BC @5:54
15 Oct 880 BC @5:54	21.283	16 Oct 720 BC @3:11
16 Oct 720 BC @3:11	19.783	16 Oct 560 BC @22:58
16 Oct 560 BC @22:58	20.150	17 Oct 400 BC @19:07
17 Oct 400 BC @19:07	22.233	18 Oct 240 BC @17:21
18 Oct 240 BC @17:21	24.850	19 Oct 80 BC @18:12
19 Oct 80 BC @18:12	26.917	20 Oct 81 @ 21:07
20 Oct 81 @ 21:07	28.283	22 Oct 241 @ 1:24
22 Oct 241 @ 1:24	29.233	23 Oct 401 @ 6:38
23 Oct 401 @ 6:38	29.667	24 Oct 561 @ 12:18
24 Oct 561 @ 12:18	28.850	25 Oct 721 @ 17:09
25 Oct 721 @ 17:09	26.717	26 Oct 881 @ 19:52
26 Oct 881 @ 19:52	24.167	27 Oct 1041 @ 20:02
27 Oct 1041 @ 20:02	21.817	28 Oct 1201 @ 17:51
28 Oct 1201 @ 17:51	20.350	29 Oct 1361 @ 14:12
29 Oct 1361 @ 14:12	19.917	30 Oct 1521 @ 10:07
25 Oct 1601* @ 23:11	22.883	27 Oct 1761 @ 22:04
27 Oct 1761** @ 22:04	25.567	30 Oct 1921 @ 23:38
30 Oct 1921 @ 23:38	27.433	1 Nov 2081 @ 3:04
1 Nov 2081** @ 3:04	28.317	4 Nov 2241 @ 7:23
4 Nov 2241* @ 7:23	28.917	6 Nov 2401 @ 12:18
8 Oct 2401* @ 3:10	27.700	10 Oct 2561 @ 6:52
10 Oct 2561** @ 6:52	26.283	13 Oct 2721 @ 9:09
13 Oct 2721 @ 9:09	23.967	14 Oct 2881 @ 9:07
average lapse:	25.62	st.dev. 3.31 hours

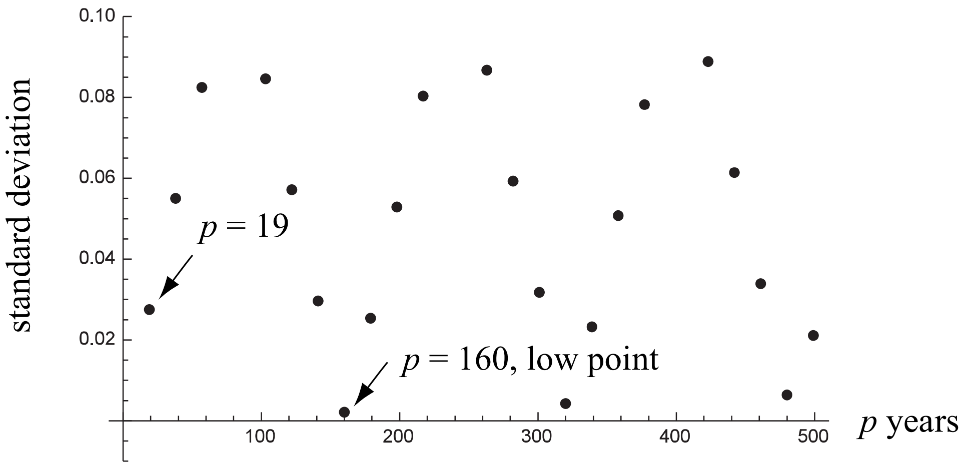


Figure 3: Standard deviation, $\sqrt{1-\cos(2\pi\sigma p)}$

Expected Value of the Variation in New Moon Dates p Years Apart

To calculate the variation between new moon years given by our earth-moon-model, recall that the *average value* of a function g over the interval $(0,\, T)$ is

$$\frac{\displaystyle\int_0^T g(x)\,dx}{T}. \tag{2}$$

Let $f(t,\, p) = w(t+p) - w(t) = \sin(2\pi\sigma(t+p)) - \sin(2\pi\sigma t)$. Since the period of $w(t)$ is $\frac{1}{\sigma}$, the average value of $f(t,\, p)$ with respect to t is 0 by (2) for any given integer p. And so the average value of the square of $f(t,\, p)$—the variation we seek—is $v(p)$,

$$v(p) = \sigma \int_0^{\frac{1}{\sigma}} \Big(\sin(2\pi\sigma(t+p) - \sin(2\pi\sigma t)\Big)^2\, dt = 1 - \cos(2\pi\sigma p). \tag{3}$$

Figure 3 is a graph of the standard deviation of the differences in moon positions p years apart as p ranges from 1 to 500. In particular, the standard deviation is lowest at $p = 160$, and is not quite so low at twice and thrice this value. Indeed, since $w(19) \approx 0.0275$ and $w(160) \approx 0.0021$ then $\frac{w(19)}{w(160} \approx 12.90$. If the ratio of the standard deviations of the difference in moon displacement from new and full phase 19 and 160 years apart in our model is positively correlated with the ratio of short and long span differences with respect to NASA's data bank, then our simple earth-moon-model has indeed anticipated the standard deviation at $p = 160$ years being significantly less than at $p = 19$ years in NASA's data. The fact that it does so is fairly remarkable, considering that the moon's position in time is chaotic! That is, our humble earth-moon-sun model is fairly powerful.

Continued Fractions and Precession of the Earth

At this point, our reaction might be, "Of course the standard deviation is less at 160 than at 19, because $\frac{59}{160}$ more closely approximates the fractional part of $\sigma \approx 12.368747$ than does $\frac{7}{19}$.

These two denominators, 19 and 160, arise from the *continued fraction* representation for σ,

$$\sigma = [12; 2,\, 1,\, 2,\, 2,\, 8,\, 12,\, 1,\, 9,\, 4,\, 1,\, 2,\, \ldots],$$

whose *convergents*—the partial expansions of this representation—approach σ, whose third, fourth, fifth, and sixth convergents of $\sigma - 12$ are

$$\frac{1}{2+\frac{1}{1+\frac{1}{2}}} = \frac{3}{8}, \quad \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{2}}}} = \frac{7}{19}, \quad \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{8}}}}} = \frac{59}{160}, \quad \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{8+\frac{1}{12}}}}}} = \frac{715}{1939}. \tag{4}$$

A second reaction to these results might be, "The average difference between long spans from Table 2 is $\bar{x}_2 \approx 25.62$ hours. Why so far from 0?"

Some of $\bar{x}_2$'s seemingly inflated value is due to chaos <u>and</u> to the *precession* period of the earth wherein earth's new year rotates around the zodiac in about 25800 years. So we could modify σ by somewhere around $\frac{1}{25800} \approx 0.0000387597$, and re-analyze our model accordingly. Alternatively and more easily—we exploit the recursive nature of the convergents in (4) and realize that the convergent $\frac{59}{160}$ is the *general mediant*, $f(n)$, of its two preceding convergents,

$$f(n) = \frac{3+7n}{8+19n} \tag{5}$$

where $n = 8$. Observe that $f(0) = \frac{3}{8}$ the third convergent of $\sigma - 12$ and $f(n)$ converges monotonically to $\frac{7}{19}$, the fourth convergent, as $n \to \infty$. Therefore, chaos and precession may have caused the relative angular velocity value σ of the moon to stray from $f(n)$ where $n = 8$. Since precession should increase σ's value which in turn means that n should decrease, this modified angular velocity has new fifth convergent either $f(7) = \frac{52}{141}$ or $f(6) = \frac{45}{122}$. Consulting NASA's data for medium spans of length 141 years and 122, we find that 141 years is the more agreeable result. For a cycle length of 141 years, Table 3 gives the mean as about 8 hours with a standard deviation of 2.51 hours. Therefore 141 bests a cycle of 160 years.

Table 3: NASA's New-moon Dates 141 Years Apart

α	hour lapse	$\alpha + 141$ years
14 Oct 3 @ 6:09	6.98	14 Oct 144 @ 11:08
29 Oct 7 @ 20:18	12.07	30 Oct 148 @ 8:22
15 Oct 11 @ 13:14	5.33	15 Oct 152 @ 18:34
31 Oct 15 @ 14:24	9.92	1 Nov 156 @ 00:19
16 Oct 19 @ 21:28	6.68	17 Oct 160 @ 4:09
2 Nov 23 @ 4:58	7.52	2 Nov 164 @ 12:29
18 Oct 27 @ 8:23	8.98	18 Oct 168 @ 17:22
3 Nov 31 @ 15:39	5.80	3 Nov 172 @ 21:27
19 Oct 35 @ 23:00	11.43	20 Oct 176 @ 10:26
4 Nov 39 @ 23:40	5.02	5 Nov 180 @ 4:41
average lapse:	7.97	st.dev: 2.51

Conclusions

The answer for our Quest of when a lunar holiday will next arrive is a toss-up between 19, 141, and 160 years, with 19 losing the standard deviation battle, yet winning a measurement test that truly matters, namely, a time lapse within man's expected lifespan. So until the human race can more than double its life expectancy of about 70, 19 beats both 141 and 160, and is the winner by default! Nevertheless, from this little study we make three observations.

- Sometimes a simple model may unveil a phenomenon's structure, such as uncovering the numbers 19 and 160 and 141—as well as remind us about pertinent related mathematical tools, such as continued fractions, (for which [3] is a good resource).
- Sometimes a simple model may suggest an important natural definition, such as the definition of a number's signature. In particular, $\mathcal{S}_\sigma$ is an explicit visual representation of the denominators of the fourth and fifth convergents of σ, and in fact, implicitly contains the rest of them as well—but that is another story.
- Even though a phenomenon's chaotic nature may dissuade us from further investigation, this model suggests that an attractor might exist within the earth-moon-sun system which somehow yanks the system into long range regularity.

Finally, we close with an advisory-alert. Since the moon is currently receding from the earth at a rate of about 3 cm/year, the angular velocity of the moon is decreasing while earth's angular velocity about the sun more or less remains constant. As time goes on the 19 year cycle of the moon will fade, and eventually wax into an 11 year cycle. The good news for our lunar holidays: we need not change our calendars for at least another four hundred thousand years.

References

[1] Aristophanes. *Complete Plays of Aristophanes*. Random House Press, NY, 1962.

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