Thinking Beautifully about Mathematics
A View of Mathematics as the Science of Measurable Orders

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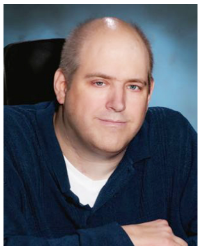

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Abstract

We initiate a view of how beauty is seen in mathematics as one that highlights the human endeavor to explore and articulate the orderly features of creation. To that end, we focus on what order is , how order is understood in our physical universe as the created order, and how mathematicians study order through the means of measuring through quantities (generally understood). It is in terms of this last way that we will define the notion of measurable order and further define mathematics as the science of them. We will describe how mathematics arises from and relates to the natural sciences by viewing the latter as aimed at an empirical understanding of the created order and the former as an intellectual endeavor that is initiated by our encounter with the created order and then discovered through abstraction. Mathematics will then be seen as pertaining to a natural science when its subject-matter can be understood with respect to a measurable order by way of abstraction. In this light, measurable orders can be viewed, in general, as abstractions from orders that (potentially) pertain to the created order.

From such a vantage point, we articulate a perspective of beauty as one that arises from the human endeavor to explore and discover the wonders found in the created order. Here we broaden an understanding of exploration and discovery to pertain to the created order in either an empirical mode, as with the natural sciences, or in a speculative mode, as with mathematics.

We close with a theological perspective of order that forms a basis for our view of mathematics as the science of measurable orders.

1 Introduction

The goal of this paper is to initiate a viewpoint of mathematics in which the subject-matter is seen as a synthesis of both being originated in human understanding and as central in the achievement of scientific advancement. To that end we will be guided in our discussion by addressing two important questions pertaining to the nature of mathematics:

1. Wigner's Problem[1]: Why is mathematics unreasonably effective to the natural sciences?
2. Ontology/Epistemology Problem [2] : Are mathematical objects invented or discovered?

[1]Eugene Wigner, "Unreasonable Effectiveness of Mathematics in the Natural Sciences", *Communications in Pure and Applied Mathematics*, Volume 13, No. I 1960

[2]For recent discussions on this problem, see: Reuben Hersh, "On Platonism." European Mathematical Society Newsletter 68 (June 2008): 17; Barry Mazur, "Mathematical Platonism and Its Opposites." European Mathematical Society Newsletter 68 (June 2008): 19-21; *Mathematics through the Eyes of Faith,* eds. James Bradley and Russel Howell, 2011: Ch. 10.

Our approach will be to address both of these questions together by articulating a broadly understood notion of **order** and seeing the natural sciences and mathematics as human endeavors to explore order in their proper contexts. In particular, we will understand the type of orders that the natural sciences explore, the orderings found in our physical world, as uniformly part of the **created order**. As such, we shall think of the physical world as ordered in a particular way, due to it being an **existing order** arising from an **act of creation** which determines both its existence and its intelligibility. Uncovering aspects of that order is a process that is initiated through the senses, in order to disclose its particular features. Understanding, though, arises as an act of intelligence which theorizes the relevant properties and relations that pertain to that order. An act of senses then, through appropriated tools, conditions, and experimentations, seek to verify, modify, or reject the hypothesis. Within this process, only certain parts and properties of the physical world are paid attention to in order to discover the pertinent objects and the laws that pertain to them within that ordering. We will want to be explicit about this mode of selective attention, which we refer to as **abstraction**, and which places the act of understanding completely within the cognitive realm. It is within this realm that order may be speculated on as it pertains to a potential created order, which we will refer to as a **cosmic order**. It is this last concept that we seek to find an explicit description of that pertains to orders that are either potential or created.

An important aspect of both theorizing and then verifying through experimentation is the way of associating quantities to properties and relations and then making measurements. In terms of orderings and the things that are ordered, quantifying them pertains to only certain aspects of their overall nature. These aspects are arrived at by abstraction and we will seek to characterize precisely those things and the ways they may be ordered which can be understood by quantities and quantifying. We will refer to such orders as **measurable orders** and define **mathematics as the science of measurable orders**. Moreover, we will hold, with Thomas Aquinas, that, to the extent a measurable order pertains to material things, the matter of the measurable order describes is strictly intelligible in nature. As such, measurable orders are not restricted to orders within creation alone, but, again, to any order that can be potentially created. In order to make that idea coherent, we need an understanding of what underlies our knowing of creation *per se* that is either potential or actual. Here we will need a sufficient account of **being** which underlies all that which does or can exist. We will see that the condition of being is not only the minimum condition for an order to be part of a creation, potential or actual, but provides the basis for understanding the minimal conditions for things to be understood as participating in an order.

With these perspectives of order in place, we may view material orders as occurring within a **hierarchy of orders**, beginning with the most particular, the created order, to the most universal, measurable orders. The levels within this hierarchy can be understood through the sciences, related by ascending via greater degrees of abstraction. For example, biology → chemistry → physics → mathematics. In terms of how this hierarchy is understood to pertain to a creation, either actually or potentially, we will identify two modes of thinking: creational thinking and thinking beautifully [3].

Creational thinking is a way of thinking of orders in two related modes. The first is the **empirical mode**, typically engaged within the natural sciences, in which the focus is on determining properties and relations within orders valid for the created order and verified via proper experimentation. The second mode is the **speculative mode** which aims to ascertain what is valid pertaining to properties of and relations among things within the order based purely upon that order's intrinsic nature. As such, there is no presumed expectation that such validations pertain to the created

[3] My thanks to Dr. Laura Smit for suggesting this last phrase.

order, but to orders that may be potentially created. Thus, for example, in chemistry, elements may be speculated to exist (beyond the current table of elements) and then determine possible ways molecules may be formed from them. Or contemplate, in physics, a physical universe classically behaving according to an inverse cube law. Ultimately, discovery in a natural science occurs when the two modes are intertwined: First, by posing theories, in the speculative mode, that serve as hypotheses for what is true of the created order. Then, in the empirical mode, such theories are verified or rejected by experimentation.

Finally, **thinking beautifully** is an enhancement upon creational thinking in which the seeking and revealing of truths pertaining to an order is made intentionally as part of our conscious awareness. Here, again, we distinguish two modes. The first, we call **(the mode of) *invenire*** is one which aims at making or enriching discoveries of objects by the selection and/or invention of tools which seek out ever richer and ever deeper properties and relations within the order the objects belong to. The second mode we will refer to as the **reflective mode** which seeks to disclose the very order by which a discovery is arrived at and made. This seeks to make clear the ways in which the human investigator, as a participant of the creator order, comes to make discoveries pertaining an order from the harmonious interplay of the acts of human senses and human understanding.

It is the aim of this paper to be the first part of two in which the concept of thinking beautifully is unveiled and made explicit in the context of mathematics. To that end, we first clarify our notion of order and distinguish between the created order and a more broader notion of cosmic order. This will then lead us to the concepts of abstraction and of creational thinking and its two modes. We will then focus on the particular type of orders relevant to the subject-matter of mathematics. Here we first clarify what we take to mean by quantity and how orders are quantified. This leads to our notion of measurable orders and of mathematics as the science of them. We then analyze how we understand thinking beautifully about mathematics in terms of the mode on *invenire*. We will reserve our analysis of thinking beautifully about mathematics in the mode of reflection to part two. Finally, we will analyze how any cosmic order, potential or actual, can pertain intelligently to a potential creation when we properly understand what it is for there to be a Creator which is the source of the existence and intelligence of that cosmos were it to be created. We will base this analysis on the theology of Thomas Aquinas and suggest how scripture supports it.

We close this introduction by noting that the ideas formulated in this paper resulted from a long engagement with and meditation upon the thought of Thomas Aquinas. In particular, the work by Bernard Lonergan in his magnum opus *Insight* [4] has been instrumental. In the sequel to this paper, we will take a closer look at how Lonergan's notion of insight provides a medium for understanding how a mathematician discovers and verifies mathematical truths.

2 Understanding creation through order

In order to get a sense of the subject-matter of mathematics as a science, we will first describe how we understand those features common to things. These are features that are perceived by our senses. It is our aim to identify those features of things generally conceived sufficient for enabling a scientific investigation, broadly understood.

To begin, we need to be specific about what it is to understand a thing as a concrete individual in

[4] Bernard Lonergan, *Insight,* University of Toronto Press; 5th edition (1992) (Hereafter *Insight*)

the world. Following the **common sense** perspective of Lonergan[5] in which we initially understand a **thing in itself** as *a concrete, whole, identity, unity which is, in relationship to ourselves, what is immediately received and understood through our senses.* In this initial awareness of such a thing, there is no effort to divorce that thing from its environment, nor sequester from it particular features from what it is in its holistic unity. Rather, the incipient understanding of that thing is as a **being**, that is, *as the objective of the pure desire to know.*[6] As such, a true understanding of that thing must conform to two laws. First, the **law of non-contradiction:** *a thing cannot simultaneously both be and not-be.* As such, any scientific claim about something in our universe that violates this law fails in some way to say something true about that thing. As Tuomas Tahko frames it: "[T]he law of non-contradiction is a general principle derived from how things are in the world. For example, there are certain constraints as to what kind of properties an object can have, and especially: some of these properties are mutually exclusive."[7] Second, **the law of excluded middle:** *a thing must either be or not-be.* This law preserves the connection between the speculative and empirical understanding: the success of an act of judgement that is made in the speculative mode ensures the success that a corresponding act of judgement can be made in the empirical mode (see next section). In other words, this law serves to bridge hypothesis and verification.

A further consequence of understanding a thing as a being, is its conformity to the **law of sufficient reason:** *everything that comes into being has a cause for its existence.* Thus scientific reasoning is movement from cause to effect, and so truths are either self-evident truths, first principles, or obtainable in an orderly way from self-evident truths, first principles and/or established truths. [8]

In this common sense first approach to understanding, we take as primary those things that inhabit the real world. As such, it takes the position that a true understanding of what a thing is is one which must be non-reductionistic, as it must first and foremost be taken as a whole unity. Thus a common sense view of a thing is to be distinguished from a scientific view, in that the latter seeks to understand that thing in terms of a particular viewpoint that reduces what it is to be to how it is to be constituted by those elements which that science takes as fundamental. From this incipient viewpoint, we understand a thing in itself to be **intrinsically ordered** as it is part of the order that is integral to creation. We will then identify this initial act to understand a thing in itself, received via common sense, as an **act of contemplation** of the intrinsic order of that thing.

A science will then be understood to traffic in a mode of understanding which views a thing as a whole reducible to certain fundamental parts (henceforth **elements**). We therefore distinguish between a thing in itself, as first understood by common sense, and the same thing as understood scientifically. In terms of the former, we will, henceforth, refer to that thing as a **created thing** and, then, use the term **thing** more generally as the subject of any mode of understanding. It is then the question of when a scientific approach to a created thing has achieved a degree of knowledge of that created thing by the way that thing is investigated by that science. The standard approach is by the method of **abduction**: the interplay of hypothesis and verification. In the end, though, the knowledge gained by that science is limited by the material assumptions and relations that that science understands how that thing is to be so constituted.

[5] *Insight* §IV.1

[6] *Insight*, §XII.1

[7] Tuomas E. Tahko, "The Law of Non-Contradiction as a Metaphysical Principle", *Australian Journal of Logic* (7) 2009, 32

[8] For more on these two laws, see W. Norris Clarke, *The One and the Many: A Contemporary Thomistic Metaphysics*, University of Notre Dame Press, 2001, pp. 19-23

To frame how we take a science to understand a thing, we define what it is to be an order. To that end, we need to be able to distinguish between what an order is within a thing and what is an order among things. To do so, we first consider Lonergan's understanding of **genus** [9]. This is a way to broadly portray how a range of things are considered in a unified way by a specific science in terms of the elements that science takes as fundamental. So sub-atomic physics considers the genus of atomic things; chemistry considers the genus of molecular things; biology considers the genus of living cellular things. In every case, a genus of things unifies those things as constituted by the pertinent fundamental elements and relations between them. In terms of the latter, these **correlations** are how the science of that genus understands and relates those things within it.

What is important in Lonergan's account is that while genera can be viewed in a hierarchy (genus of cellular things can be studied in terms of the genus of molecular things which, in turn, can be studied in terms of the genus of atomic things) the correlations of a higher genus are neither independent of nor reducible to that of a lower genus. Thus while things in a higher genus can be in certain ways described by things of a lower genus, those things in the lower genus will not be found as things in the higher genus.

Now, we will define an **order** to be an organization of things in accord with the relevant elements and correlations that constitute the formation of a genus. An **order within** a given thing will refer to how it can be in its particular genus in accord with things as constituted by the elements and their correlations coming from lower genera, in the hierarchy, for which that particular genus is at the apex. From that perspective, that thing is understood to **participate** in that genus or order. An **order among** given things is the totality of correlations that pertain to those given things that are understood as participants of that genus.

Finally, we use the term **created order** to refer to our own physical universe with its hierarchy of orders and define a **cosmic order** to be the view of a cosmos together with its particular hierarchy of orders that arises from speculation about how our universe could be.

3 Abstraction and creational thinking

In order to come to how we will understand the subject-matter and science of mathematics, we will describe the way humans understand and reason about cosmic orders. What is foremost about delineating such a description is the recognition that any disclosure of truths pertaining to an order is initiated in the senses, but is only finally understood in the mind. This transition is accomplished through the process of abstraction to which we need to give careful attention.

We begin by following Thomas Aquinas [10] to define **abstraction** as the "operation ... by which in understanding what a thing is, it distinguishes one thing from another by knowing what one is without the other, either that it is united to it or separated from it". In our particular context, it is by way of abstraction from things of our experience that orderings of those things (that is, their genera and its elements and correlations) are discovered and determined. Furthermore, in arriving at an understanding of things in terms of a particular genus, we need to only understand first how each are ordered within, with respect to that genus. As such, those aspects that are not pertinent to understanding a thing within its particular genus (for example, particular time, particular place)

[9] *Insight* pp. 280ff

[10] Armand Mauer, *Thomas Aquinas: The Division and Methods of the Sciences*, (Mediaeval Sources in Translation) Pontifical Institute of Mediaeval Studies; 3rd ed. (1963), pp. 28-31

are abstracted away. Further, those aspects that remain and are pertinent to the genus it inhabits, along with the relevant ordering, are universal with respect to that genus.

Scientific investigation within the created order begins by understanding that order as pertaining to a cosmos. From such a standpoint, investigation of things is initially conducted in abstraction. As such, classical laws pertaining to things are first understood as pertaining to some cosmos. Discoveries then arise and are understood as perceived within that cosmos and with respect to its particular order. To ascertain whether the created order conforms to that cosmic order, investigations are conducted, through particularly crafted experiments, in which the required abstract conditions are sharply fulfilled, to obtain a verification. We will take this perspective on the practice of science as a special case of **creational thinking**. In particular, we will take the activity of the scientist as a creational thinker to be one in which hypotheses are proffered about the created order, but initially understood to pertain to some cosmos. In this way, investigation is conducted within that science's particular theoretical framework. The scientist may then endeavor to determine whether the resulting consequences obtained are valid to the created order by empirical means via, for example, experiments within a laboratory.

Thus, in general, creational thinking aims to understand ways our universe may be organized as a cosmic order. The initial stance is to engage in creational thinking in the **empirical mode**. This particular type of creational thinking requires the common sense elements of experimenting, testing and verifying theories conducted within the physical realm. The development of those theories, though, involve engaging in speculations by posing such questions as "Is it the case . . . ?" or "What if . . . ?". This is the **speculative mode** of creational thinking and is involved in any effort to understand any ordering that can take possibly place in our cosmos. For example, such theoretical models in physics, such as superstring theory, or, more generally M-theory, is well developed speculatively, but still seeks experimental verification. Such a theory may be viewed as creational thinking pertaining to a cosmic order which physicists, in the empirical mode, are seeking to establish whether that order is (relevant to) the created order.

By ordering creational thinking so that the empirical mode is first, knowledge begins with our sense reception of the world and our common sense understanding that results. Subsequently, understanding the world more deeply requires the speculative mode by abstracting in things what it takes to theorize within the particular scientific mode and testing the theories produced within (for example) a laboratory to verify, modify, or reject. The speculative mode can also be engaged to postulate theoretical frameworks for orders and produce theories that pertain to such orders without any immediate requirement that they be found relevant to the created order via the empirical mode. Here, things may simply be understood as being potentially created things by simply presuming them to possess being and test their potential existence through the scientific theories themselves. Ultimately, though, it is only in the empirical mode that speculation may begin and it is only a return to that mode that the pertinence of those theories to the created order may be tested for their validation.

As we continue to set out how we perceive the concept of order and what constitutes a science of order , we note again that orders can be viewed in a hierarchical fashion in which one order found on a higher level will be common to all the orders found on lower levels. For example, we have the following hierarchy

$$\text{living things} \to \text{molecular things} \to \text{atomic things} .$$

Each level has their orders within and among, but also may be understood via the higher levels. In each level, there is a corresponding science: biology, chemistry, and physics. A science of each level may be conducted independent of the lower levels and each science may be conducted either speculatively or empirically. Finally, we note that any and all such hierarchy of orders are understood level-wise and together as bound by the condition of potentially or actually having being.

We now aim to concentrate on the type of orders that are potentially common to all possible orders, namely orders of things understood through quantity alone. Such an order is called a measurable order and mathematics as its science.

4 Quantity and the quantified

We now look at a specific type of order, one that we claim every other order participates in, namely order that is understood through **quantity**. Thus, to frame quantity in a way to speak to order in the broadest sense, we first identify two characteristics of quantity that we will take as fundamental at the basic level:

I. That which can be increased or diminished (L. Euler [11])

II. That by which things are inclined, distinguished, and limited. (Bonaventure [12])

The first of these is straightforward and expected in how we initially conceive quantity. The second brings to quantity the way equality works by, for example, the requirement that quantities associated to equal things must be equal. We further codify this utilizing Euclid's Common Notions, which we refer to as **basic rules of calculation**:

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole is greater than the part.

Taken together, these propositions provide not only the first principles for being a quantity, but also for a thing to be **quantified**. Furthermore, to be clearer on this last point, we shall stipulate the conditions under which quantifying occurs. To obtain such conditions, enough to be sufficiently universal, we follow James Franklin [13] by requiring:

A. Parts unified into a whole.

B. Intrinsic notions of same and difference.

[11] Leonard Euler, *Elements of Algebra*, Springer; 1972 edition

[12] Saint Bonaventure, *Journey to the Mind of God*, Translated by Oleg Bychkov, I.11

[13] James Franklin's *An Aristotelean Realist Philosophy of Mathematics*, Palgrave Macmillan; 2014 edition

Together these two conditions determine a **structure**. In keeping with our current theme, we will define a **measurable order** to be a structure together with the parts quantified by the **rule of invariance:** *two parts that are the same have equal quantities.*

Before we examine how to define mathematics in terms of measurable orders, we first need to identify what constitutes the range of things that can be counted as quantity. Here we primary follow Lonergan by taking an **ordinal** perspective of quantity and note that a **cardinal** perspective is also equally important for framing quantity, but we will forgo an analysis of that viewpoint.

We begin with the **natural number system**:

$$0 \quad < \quad 1 \quad < \quad 2 = 1+1 \quad < \quad 3 = 2+1 \quad < \quad 4 = 3+1 \quad < \quad \ldots$$

Implicit in this description is the sufficiency of 0, 1, +, =, and $<$ to generate this number system along with the "$\ldots$" to indicate the **well-ordering principle** at work to provide this system with it's potentially infinite completeness. Additional operations of $\times$ and exponentiation can be further defined.

Next, inverse versions of the operations (difference, division, roots, logarithms) can be constructed and properties determined, but are limited in when they can be defined. These limitations can be overcome by defining new numbers from these operations (rational numbers, irrational numbers, imaginary numbers) limited only by the rules of non-contradiction (e.g. no division by 0) and sufficient reason. This in turn leads to new number systems (rationals, reals, complex numbers) each having operations and rules of calculation under which they operate. Furthermore, the rules enable the operations to act on numbers in an indeterminate fashion, allowing the introduction of variables, and leading to polynomials, rational functions, and, eventually, functions in general.

This process of developing new systems from old by the introduction of new operations and constructions that require, in turn, the introduction of additional new objects and the modification of rules for the operations and constructions is the process of producing (after Lonergan) **higher viewpoints** [14]. From this concept, we will define a **general quantity** to be any term found in a system that is constructed as a higher viewpoint of the natural number system. We note that general quantities, either of specific or abstract type, are themselves measurable orders. Moreover, we further modify our definition of measurable order to allow for quantifying via general quantities. Thus, we may finally we make the following definition: **mathematics is the science of measurable orders.**

5 Measurable orders from cosmic orders

We now compare our concept of measurable orders with the notion of cosmic order. In particular, we show that any cosmic order has the capacity to be understood at a certain level as a measurable order in a way consistent with that particular cosmic order. To begin, we first observe that any thing viewed as participating in a particular cosmic ordering must, in a primary way, participate in being. Following Aristotle, as delineated by Thomas Aquinas, we can understand how things participate in being in terms of a proper order:

- First as being.

[14] *Insight* pp. 40ff

- Second in terms of quantity.
- Third in terms of sensible qualities.
- Fourth in terms of particular place and time.
- Fifth in terms of being subject to change.

Now, understanding a thing in terms of this hierarchy at a particular level depends on understanding that thing in terms of earlier levels. From the perspective of the matter that constitutes the thing, the last three levels are understood in terms of **sensible matter**. This is not so for the first two levels which are understood independent of the subsequent levels and so are, together, understood in terms of **intelligible matter**. From this, we draw two conclusions. First, quantity and the quantifiable (and so, measurable orders) are materially constituted in terms of intelligible matter. Moreover, understanding a thing in terms of intelligible matter is arrived at by abstracting from the sensible matter in the thing. We will refer to this manner of abstracting intelligible matter in a thing as **formal abstraction**. Second, any thing viewed as participating in a particular cosmic order will do so in terms of some aspect of its sensible matter. Scientifically, such aspects must belong to the sensible qualities in order to locate classical laws that are empirically verified. Furthermore, the elements and correlations of that order will also be understood in terms of those sensible qualities. By formal abstraction, the thing in terms of those elements and correlations can be quantified. From that, a measurable order is determined by abstracting the intelligible matter in those elements and their correlations. Thus, measurable orders serve cosmic orders by:

- interpreting correlations in terms of identities,
- identifying when correlations hold, and
- giving limits to when correlates and/or correlations may obtain.

We therefore conclude that a scientific perspective of a cosmic order is one that can be understood through measurable orders that potentially arise from abstractions. In turn, those cosmic orders that can be understood and analyzed by a particular measurable order will be ones that fit a **model** of that measurable order through abstraction. Thus, the science of measurable orders provides a context to study any cosmic order. Finally, Wigner's Problem is addressed to the extent that that science discloses an understanding of the created order by speculation of the physical universe as a cosmic order. Then the particular order that is the purview of that science can be framed within a measurable order once things within that scientific genus, in terms of elements and correlations, are properly quantified and modeled.

6 Thinking beautifully about mathematics

We now initiate a look at the shape of creational thinking in regards to the science of mathematics. To that end, we will describe a perspective on the structure of mathematical knowledge as it pertains to the search for and classification of mathematical objects.

The aim to understand mathematical objects and provide a classification of them raises the Ontology/Epistemology Problem. From that perspective, we will consider the elements of what is part of invention. In particular, we need to clarify the ways the human will plays a role in navigating a path toward making a discovery. In particular, forming questions, selecting tools, reviewing prior research, making calculations, performing experiments, forming and proving propositions, etc. We will then examine how objects are defined, analyzed, and understood as existing. This will lead us to addressing in what ways the role of discovery is at play. In the end, we will understand these two objectives as unified within the **mode of *invenire*** when thinking beautifully about mathematics.

We begin by describing how objects are defined. A **mathematical object** will be understood to be *a thing that is constituted simply by intelligible matter alone.* Such objects can then be understood within mathematics through a **field of mathematics**: a framework which involves prescribing either

- a type of general quantity through formally specifying definitions and rules of combination and relations, or
- a type of structure by formally defining how basic parts are understood, how such parts are understood to be formally the same or different, and by formally identifying postulates that stipulate how parts are to be constructed or related within the whole.

We will say that a field of mathematics is **developed** through the introduction of operations and constructions, to produce further objects, as well as the use of those processes that lead to higher viewpoints. Furthermore, development also involves the production of modes of quantification that connect (systems of) general numbers to structures to form measurable orders.

Now, mathematical objects need not be understood solely with respect to a single field of mathematics. In fact, distinct fields of mathematics may have a non-trivial overlap, or may be viewed in a hierarchical fashion. Thus objects may be studied from the vantage point of different fields of mathematics if one or more scenarios like this hold. Furthermore, an object may be framed in two different ways so as to be studied and understood in terms of two different fields of mathematics. In fact, true mathematical objects are not determined a priori by any field of mathematics, but rather by framing an object within a particular field with properties and relations disclosed and understood in terms of the framework and language of that field. Thus it is here that the tension between invention and discovery can be seen. The erection and development of a field of mathematics requires forming questions, the articulation of definitions, the selection of postulates, and the choices of symbols and language to frame objects to enable the discourse and exploration of those objects within that particular field (making calculations, stating and proving propositions, . . .). These may be viewed as **acts of invention** by the mathematician. Additionally, these acts include the posing of questions to be explored, the gathering of evidence, the formation of hypotheses, and, of great importance, the ***design** of frameworks and methodology.* **Discovery of objects** then can occur by understanding them sufficiently as framed within a field, articulating or specializing properties which characterize a potential (type of) object within that field, testing and refining the hypotheses as regard to them, and, finally, drawing the conclusions that give a deeper understanding.

But what of mathematical objects themselves? First, we will say, like things, that an **object** is a being which is a unity, identity, whole. We take such an object as midway between a created thing and a mathematical object. In this way, a mathematical object is an object that can be understood through the lens of (a field of) mathematics by abstraction. And so, we will say that our understanding of mathematical objects occur by **acts of discovery** understood as the revealing of the properties and relations those objects are seen to possess through a framing within mathematics. Furthermore, once these objects are understood, further acts of invention seek to speculate on ways to deepen the understanding of such objects and the relationships between them, opening up to new acts of discovery.

But what is the state-of-being of objects? We will take them to *arise from abstractions of created things by speculation*, but only to be considered as beings-to-be-understood by mathematics without committing them to be the subjects of any particular mathematical field for analysis. From this perspective, the object is understood by a mode of abstraction that is prior to a formal abstraction. In that manner, such an object may be considered as being participated by things as subject to a cosmic order in general, or participated possibly by created things. In the latter situation, it is by experimentation that created things are verified to be an object by participation. If such is the case, we say that such a created thing is a **similitude** of the object. For example, a chemist can understand water in a lake as the similitude of molecular objects ordered within by the molecule H_2O. Or speculate that a created thing understood as ordered within by atomic point particles be considered, instead, as ordered within by string particles, i.e. closed loops curled up in tightly curved space existing in higher hidden dimensions. Here, we have string objects which are increasingly well understood in the cosmic order governed by string theory, but there has yet been any verification that created things are similitudes of string objects.

Thus the status of objects are on par with the status of things as potentially created. These are dependent on the status of cosmic orders in relationship to the created order. Hence, the first test that properties pertaining to a mathematical object potentially reflect properties of an object is their persistent conformity to the law of non-contradiction. That is, an understanding of a mathematical object reflects an understanding of a thing within a cosmos insofar as that object arises from abstracting that potential thing. Hence any knowledge of a mathematical object provides knowledge of a thing in a cosmos which can be understood as a similitude of that object. In the next section, we indicate in what way this last way of understanding is grounded in the divine activity of the Creator.

We close this section by describing ways acts of invention and discovery can play out in the research of a mathematician. Here we follow the approach of David Mumford [15] to identify such categories. [16] Before outlining them, we summarize the overall aims of these modes of research in terms of two objectives: the **construction and analysis of tools** (acts of invention) for the purposes of **identification and classification of objects** (acts of discovery). Within this context, mathematicians can be divided into the following modes of researchers:

- Explorers: These are seekers of objects with particular or special properties. These need not be initially understood within a particular field of mathematics, but make use of particular fields in order to frame the motivating questions and build and/or utilize the tools to better understand those objects. These are often sought in terms of their properties and the relations between them and to other types of objects, and, ultimately, classify them. In the end, some seek special types of objects with unusual properties (gem collectors) or give an organization to objects of a particular type into a novel ordering (mappers). Examples of the mathematics

[15] David Mumford, **Math & Beauty & Brain Areas**, http://www.dam.brown.edu/people/mumford/blog/2015/MathBeautyBrain.html

[16] A similar analysis of Mumford's categories as understood in terms of St. Bonaventure's theology is given by the author in "Seeing Beauty in Mathematics: Reflections on Bonaventure's "Reduction" of Mathematics to Theology", Chithara 57 (1) pp. 38 – 57

discovered by explorers include:

  - Pythagoras's Theorem arising from the study of many tables of data produced from surveying efforts in ancient Mesopotamia.
  - Theaetetus and Schlafli's determination of regular polytopes in all dimensions.
  - Cantor's exploration of higher infinities in set theory.
  - William Thurston's program to classify 3-manifolds.
  - The classification of finite simple groups.
  - Michael Artin's adaptation of the theory of algebraic geometry to construct and study varieties associated to non-commutative algebras.

- Alchemists: These are mathematicians who find connections between two areas (not necessarily fields) of mathematics that have not been previously connected. Here, areas of mathematics can be less precise than fields of mathematics and may be prescribed in terms of conditions or properties on objects that can be interpreted in more than field. This can lead to a synthesis of two existing fields or a founding of a brand new field. Examples of the mathematics discovered by alchemists include:

  - Descartes's efforts to connect geometry to algebraic equations to form analytic geometry.
  - DeMoive's formula: $e^{i\theta} = \cos\theta + i\sin\theta$
  - Oscar Zariski's initiating the use of the tools of commutative algebra to address difficult questions in algebraic geometry.

- Wrestlers: These are mathematicians who are focused on the quantifying of objects within a measurable order and "[focus] on relative sizes and strengths of this or that object; [they] thrive not on equalities between numbers but on inequalities, what quantity can be estimated or bounded by what other quantity, and on asymptotic estimates of size or rate of growth". Examples of the mathematics discovered by wrestlers include:

  - Hardy and Ramanujan's formula for numbering the partitions of an integer.
  - Riemann and Hademard's work on the Prime Number Theorem
  - Zhang's work in estimating the sizes of gaps between prime numbers.

- Detectives: These are researchers who tackle questions that have shown to be deep and difficult to answer, but who, nonetheless, seek clues, follow trails, and attempt alternative perspectives, confident that a solution can be found. Some (Strip miners) seek to "[uncover] a hidden layer underneath a visible superficial layer in order to solve a problem. The hidden layer is often more abstract." Others (Baptizers) invent "[a name for] something new, making explicit a key object that has often been implicit earlier but whose significance is clearly seen only when formally defined and given a name." Examples of the mathematics discovered by detectives include:

  - Eudoxus and Archimedes's explorations in the nature of number in understanding and estimating their occurrences in geometry and the physical world giving the beginnings of the theory of the real number system.
  - Grothendieck's program revolutionizing algebraic geometry.
  - Andrew Wiles's work to prove the Taniyama-Shimura Conjecture.
  - Egorov and Luzin's work on the Continuum Hypothesis by naming and investigating sets that may violate it (thereby launching the field of descriptive set theory).

- Naming of π as the ratio of circumference and diameter of the circle
- Naming the number e so that $\frac{d}{dx}\, e^x = e^x$.
- Naming of i as the "number" that satisfies $i^2 = -1$.

Such approaches to engaging in mathematical research see the search for mathematical objects as being prior to any framing of them via fields of mathematics. Thus the aim to identify objects and disclose their properties is initiated by identifying enough of their structural properties (possibly within a particular field of mathematics) to quantify the parts and, hence, view those objects within the context of a measurable order. This is a move from being to quantification that enables it to be understood in a formal way and so be understood within a field (or fields) of mathematics. This is a way of thinking beautifully about mathematics which can be seen as a marriage of invention and discovery. In fact, following David Bentley Hart, we will define this marriage as an **act of *invenire***: "The Latin *invenire* means principally 'to find', 'to encounter', or (literally) 'to come upon'. Only secondarily does it mean 'to create', or 'to originate'" and so "every genuine act of human creativity is simultaneously an innovation and a discovery, a marriage of poetic craft and contemplative vision that captures traces of eternity's radiance in fugitive splendors here below by translating our tacit knowledge of the eternal forms into finite objects of reflection, at once strange and strangely familiar." [17]

7 Cosmic orders as divine ideas in the Creator

We end this paper by giving a theological view on cosmic orders as grounded in the divine activity of God. Our primary source is the medieval theologian Thomas Aquinas.

First, Aquinas observes (Summa Contra Gentiles 3.97-8) that the primacy of a thing's existence over what a thing is is in its being. Here we take first the act of being (*esse*) that brings a thing into existence (*ens*) as determining that thing's essence. The *esse* is then the act in accord with the intended form that brings the pure potency of matter from non-being to being as a thing with an essential nature. Thus *esse* is the ground of all forms and all things that come to be come as a proportion of *esse* and essence. As such, the source of *esse* is one in which *esse* and essence are at once one and the same and thus is referred to as *ipsum esse subsistens* (subsistent act of being itself) which we may take to be God, who is his own existence pure and simple. And so by having form, a thing exists and by existing it resembles God, so that form is nothing else than God's resemblance in things. Now things can resemble something absolutely simple by closeness or remoteness, things more closely resembling God being the more perfect, so forms differ by degrees of perfection. Thus variety in forms requires different levels of perfection. Variety in things requires an order in levels in things and mere equality.

Next, Aquinas shows that God's providence orders everything to a goal - his own goodness - not as if what happens can expand that goodness, for things are made to reflect and express that goodness as much as possible. Created things must all fall short of the full goodness of God. So in order that things may reflect that goodness more perfectly there is a variety of things for which what one thing couldn't express perfectly could be more perfectly expressed in a variety of ways by a variety of things. In other words, any form (e.g. owl-ness, cat-ness, rock-ness, sphere-ness) that God intends

[17] David Bentley Hart, "A Perfect Game", reprinted in *A Splendid Wickedness and Other Essays* Eerdmans Publishing 2016 p. 44

to be part of creation has an unlimited way of expressing itself as that form. As a created thing is a finite material expression of a form, that form is expressed in that thing in a limited way and, hence, has only a finite portion of the power to act in terms of that form as possessed by that form. On the other hand, the full expression of that form can be seen by the variety of ways and by the variety of things that form is the expression of materially, in creation, and throughout time. All those things that do express that form can be seen as being similar by that expression, but no two express that form identically.

Thus, the perfect goodness that exists one and unbroken in God can exist in creatures only in a multitude of fragmented ways. Variety in things comes from different forms determining their species. Because of their goals things differ in forms. Hence there is an order among things. Thus in the hierarchy of reasons behind God's providence, we have:

- God's own goodness: the ultimate goal which first starts activity off.
- The many-ness of things: determined by different degrees of forms and matters, agents and patients, activities and properties.

Aquinas notes that while God must love his own goodness, it doesn't follow necessarily that creatures must exist to express it, since God's goodness is perfect without it. The coming to be of creatures finds its first reasons in God's goodness and depends on a single act of God's will. The reason for the variety of creatures is because God does will to share his goodness as far as that is possible by way of resemblance to God. This does not necessitate this or that measure of perfection or this or that number of things. Thus a reason for a thing to have this form or that matter is by God through an act of will, deciding the number of things and the measures of perfections. Furthermore, as acutely analyzed by Fran O'Rourke: [18] "The beauty of the universe consists in the harmony, proportion, order, and mutual solidarity of beings which are infused with a single desire for their unique and universal end" [19] for

> ... divine beauty is the cause and goal of creation. Out of love for his beauty God wishes to multiply it through the communication of his likeness. He makes all things, that they may imitate divine beauty. Aquinas is thus able to declare: 'The beauty of the creature is nothing other than the likeness of divine beauty participated in things ... Created being itself (*ipsum esse creatum*) is a certain participation and likeness of God.' The beauty of the creature is its very being ... Each being is a participation in the divine beauty, an irradiation of the divine brilliance. [20]

God's providential ordering of the universe speaks to the nature of mathematics via a perspective of created things as essences formed by an act of existence which is something divine in things. [21] In terms of the ordering of things, we take the *mathematical objects as intelligible species that are ordered in a measurable way*[22]. These are arrived at by abstracting from the order experienced in the physical world. In particular, we may take the items of the world as the existing things for which their variety in unity is undergirded by the analogy of existence (*esse*). This holds for all things, especially necessary truths, such as found in logic and mathematics. For here, these truths,

[18] Fran O'Rourke: *Pseudo-Dionysius and the Metaphysics of Aquinas*, University of Notre Dame Press, 2005
[19] Cf. 273
[20] Cf. 273-74
[21] Here we follow David Burrell's *Knowing the Unknowable God* (University of Notre Dame Press (2001)) pp. 66-70
[22] emphasis mine

as pertaining to created things, are, in God, “divine ideas" (*logoi*): not by which God understands, but that which God understands.

> “God’s knowing is not propositional and God’s creation is not to be considered literally after a craftsman who must look at blueprints, or be thought to work from an idea. Rather, God in knowing his essence as imitable in this particular way by this particular creature, knows his essence as the nature and idea proper to that creature." (Summa Theologica I.15.2).

The “ideas" can be likened in God to the “form in the mind of the builder" so that for a creation to be structured in such a way is for that created nature to imitate the divine nature. Thus God does not create these ideas, but in creating “they express the structure creation will assume as God’s creation - ‘imitating the divine essence’". The ideas then depend upon God’s nature and not on God’s will and are in the mind of God in an exemplary way by understanding what must be the case if the divine essence is to be initiated by such a nature. Thus the ideas act as “rules of inference, or constraints of matter, purposes and agents in building something" in which the world can be taken to be structured in such a way as to be open to scientific inquiry, not by being made of such formal features, but by its constituent parts yielding to intellectual inquiry through those features.

Thus the nature of mathematics, as pertaining to the abstract structures reflecting the formal features in creation, is grounded in God’s intentional act in ordering the universe: for “in every effect, the ultimate end is the proper intention of the principle agent . . . the highest good is the good of the order of the universe . . . therefore the order of the universe is the proper intention of God" (Summa Theologica I.15.2). The “divine ideas" are thus instrumental to God’s making, being distinct from God as he is utterly first, creating by giving *esse* and intending the “good order of the universe," so that all emanates from God. Thus the truths of logic and mathematics become instrumental to creation as the necessary condition of the divine nature being imitated in such a way. So in order for there to be such a creation, it must issue in such a matter that these truths share in as part of the hypothetical necessity proper to contingent being. Thus, as pertaining to the divine act of creation, mathematics understands not only quantifiable created things, but also quantifiable things in any cosmic ordering of a cosmos.

We close by giving a scriptural perspective of this theological account of cosmic orders, and our understanding of them, in a Trinitarian key.

> In the beginning God created the heavens and the earth. The earth was without form, and void; and darkness was on the face of the deep. And the Spirit of God was hovering over the face of the waters. Then God said, “Let there be light"; and there was light. And God saw the light, that it was good; and God divided the light from the darkness. God called the light Day, and the darkness He called Night. So the evening and the morning were the first day. – Genesis 1:1-5 (NKJV)

> The Lord possessed [Wisdom] at the beginning of His way, before His works of old. I have been established from everlasting, from the beginning, before there was ever an earth. When there were no depths I was brought forth, when there were no fountains abounding with water. Before the mountains were settled, before the hills, I was brought forth; While as yet He had not made the earth or the fields, or the primal dust of the

world. When He prepared the heavens, I was there, when He drew a circle on the face of the deep, when He established the clouds above, when He strengthened the fountains of the deep, when He assigned to the sea its limit, so that the waters would not transgress His command, when He marked out the foundations of the earth, then I was beside Him as a master craftsman; And I was daily His delight, rejoicing always before Him, rejoicing in His inhabited world, and my delight was with the sons of men. – Proverbs 8:22-31 (NKJV)

Wisdom reacheth from one end to another mightily: and sweetly doth she order all things ... O God of my fathers, and Lord of mercy, who hast made all things with thy word, and ordained man through thy wisdom, that he should have dominion over the creatures which thou hast made, and order the world according to equity and righteousness, and execute judgment with an upright heart ... For thy Almighty hand, that made the world of matter without form ... thou hast ordered all things in measure and number and weight. – Wisdom of Solomon 8:1; 9:1-3; 11:17a, 20b (Apocrypha, KJV)

In the origin there was the Logos, and the Logos was present with God, and the Logos was god; This one was present with God in the origin. All things came to be through him, and without him came to be not a single thing that has come to be. In him was life, and this life was the light of men. And the light shines in the darkness, and the darkness did not conquer it ... It was the true light, which illuminates everyone, that was coming into the cosmos. – John 1:1-5, 9 [23]

The four living creatures, each having six wings, were full of eyes around and within. And they do not rest day or night, saying:

“Holy, holy, holy,
Lord God Almighty,
Who was and is and is to come!"

Whenever the living creatures give glory and honor and thanks to Him who sits on the throne, who lives forever and ever, the twenty-four elders fall down before Him who sits on the throne and worship Him who lives forever and ever, and cast their crowns before the throne, saying:

“You are worthy, O Lord,
To receive glory and honor and power;
For You created all things,
And by Your will they exist and were created."

– Revelation 4:4-11 (NKJV)

God, in creating, creates by first understanding fully and in unity the universe that He will create. This understanding is, in God, an act of artistry that the Father gives to the Son, the Logos, to design by giving creation its creative meaning and form (the divine ideas). This is the light that

[23]David Bentley Hart, *The New Testament: A Translation*, Yale University Press (2017) 168

gives creation its order and the means for humans to see that order and thereby come to understand it in its possibilities and actualities. It is the Spirit that proceeds from the Father and the Son as the act of being, carrying out God's will, by which creation is *ex nihilo* in accord with the meaning and form the Son has designed and the Father wills to be. It is also the Spirit that reveals the purposes of creation to mankind that are in the Father and the Son.[24] In part 2 to this paper, we will examine the reflective mode of thinking beautifully by further exploring the mathematician's aim to know truths in mathematics and the means to fulfill that aim framed within a theological conception of illumination and Lonergan's notion of insight.

[24] 1 Corinthians 2:6-16 (NKJV)