Maximum Elements of Ordered Sets and Anselm's Ontological Argument

Doug Ward (Miami University, Ohio)

Doug Ward (B.A., Haverford College; M.S., Carnegie-Mellon; Ph.D., Dalhousie) is Professor of Mathematics at Miami University, where he has taught since 1984. His research is in variational analysis and the mathematics of continuous optimization problems. He enjoys teaching courses in calculus, optimization, and the history of mathematics. He also writes about history and the Bible, and his work can be accessed at http://graceandknowledge.faithweb.com/

Abstract

I present a theorem about a set containing a maximum element with respect to some asymmetric ordering. This theorem aims to elucidate Anselm's ontological argument, a classical proof of the existence of God.

1 A Theorem on Ordered Sets

In this note I will state and prove a simple mathematical theorem about a set endowed with an ordering. This theorem is very easy to prove, but it has a significant application.

I will begin by establishing some basic notation. Let T be a set. I will employ the usual notation $x \in T$ to indicate that x is an element or member of T. If x is not an element of T, I will write $x \notin T$. For a set E, I will say that E is a subset of T, denoted $E \subset T$, if every element of E is also an element of T. I will denote by $T \setminus E$ the set of elements of T that are not elements of E.

I will denote by $T \times T$ the Cartesian product of T with itself; that is, $T \times T$ consists of all ordered pairs (x, y), where x and y are elements of T. I will refer to any set of ordered pairs R, where $R \subset T \times T$, as a *relation on* T.

Now let R be a relation on T. We can think of R as an *ordering on* T by writing $x \succ y$ ("x is better than y") whenever $(x, y) \in R$. I will say that R is asymmetric if whenever $(x, y) \in R$, then $(y, x) \notin R$. In the notation of orderings, the ordering $\succ$ is asymmetric if whenever we have $x \succ y$, then we do not have $y \succ x$.

I will say that $m \in T$ is a *maximum element of* T *with respect to* $\succ$ if $m \succ y$ for all $y \in T$ with $y \neq m$. Notice that if $\succ$ is asymmetric, then maximum elements are unique. (If m_1 and m_2 are maximum elements with $m_1 \neq m_2$, then we have both $m_1 \succ m_2$ and $m_2 \succ m_1$, violating the assumption that $\succ$ is asymmetric.)

To illustrate these definitions, suppose that $T = \mathbb{R}$, the set of all real numbers, and let R be the set of ordered pairs of real numbers (x, y) such that $x > y$. Then R is an asymmetric relation, and $x \succ y$ simply means $x > y$. So, for example, $(5, 3) \in R$, since $5 > 3$, but $(3, 5) \notin R$.

In this example the set T has no maximum element, since there is no largest real number. On the other hand, suppose that we let T be the set of all real numbers along with an additional element ∞ (infinity) having the property that $\infty > y$ for all real numbers y. This new enhanced set, which I will call the set of extended real numbers, does have a maximum element: ∞

Here is another example. For real numbers a and b with $a < b$, Let $[a, b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$, and let $T = [a, b] \times [a, b]$. For elements (x_1, y_1), (x_2, y_2) of T, define $(x_1, y_1) \succ (x_2, y_2)$ to mean that $x_1 > x_2$ and $y_1 > y_2$. Then T contains the maximum element (b, b).

The ordering in this second example is known as the *coordinate-wise ordering.* Notice that there are many pairs of elements of $\mathbb{R} \times \mathbb{R}$ that cannot be compared using the coordinate-wise ordering. For instance, when $x > 0$ and $y < 0$ or when $x < 0$ and $y > 0$, then neither $(x, y) \succ (0.0)$ nor $(0, 0) \succ (x, y)$ is true.

I can now state a theorem about sets that have a maximum element with respect to some asymmetric ordering.

Theorem: Let T and E be sets such that $E \subset T$. Let $\succ$ be an asymmetric ordering on T such that

1. T has a maximum element m with respect to $\succ$;
2. for each $y \in T \setminus E$, there exists $x \in E$ such that $x \succ y$.

Then $m \in E$.

The theorem is easy to prove. Suppose that $m \notin E$. Then $m \in T \setminus E$, and by assumption 2, there exists $x \in E$ such that $x \succ m$. Now by assumption 1, we also have $m \succ x$, contradicting our assumption that $\succ$ is an asymmetric ordering. Since the assumption that $m \notin E$ leads to a contradiction, it must be that $m \in E$.

As initial illustrations of the theorem, consider the aforementioned examples. In the case where T is the set of extended real numbers, the set E can be any subset of T that includes the maximum element ∞. In our second example, where $T = [a, b] \times [a, b]$, E can be any subset of T that contains (b, b).

2 An Application in Philosophy

My motivation in formulating the theorem is a particular application in philosophy. In the corollary below, let T be the set of all things that can possibly be imagined, and suppose that T is endowed with an asymmetric ordering $\succ$. Let E be the set of all things that exist in reality, where we assume that E is a subset of T. Then we have the following result:

Corollary: Suppose that T has a maximum element M with respect to $\succ$. Suppose that for any thing imaginable that does not exist in reality, something exists in reality that is better than that thing. Then M exists in reality.

This corollary is a simple form of the ontological argument, a classical proof of the existence of God first proposed by Anselm of Canterbury (1033-1109 AD) in his *Proslogion* (1078 AD). If God can be defined as the best thing imaginable, and if every nonexistent thing is surpassed by something that exists in reality, then God (the maximum element M in the corollary) exists in reality.

Anselm was a Benedictine monk who later served as the Archbishop of Canterbury. He wrote *Proslogion* as an exercise in "faith seeking understanding." His goal was to present an argument for God's existence that would lead to a greater understanding of, and appreciation for, God's attributes.

Anselm defined God as "that than which nothing greater can be conceived," a definition consistent with his Christian beliefs. In both Christian and Jewish theology God is unique as the Creator of everything, and he is greater than all of creation (see for example Isa 44:6,24; 45:22-23).[1] God's status as Creator and Ruler of all makes him a "maximum element" among all the things that can be conceived.

Anselm's argument was controversial from the start. The argument was first challenged by a contemporary, fellow Benedictine monk Guanilo of Marmoutiers. Guanilo posed the following question: Suppose we think of a kind of "fantasy island," the greatest, richest, most beautiful island imaginable. By the reasoning used in the ontological argument, would not such an island have to actually exist? I would answer Guanilo's question by suggesting that the set of all fantasy islands, like the set of real numbers, will have no maximum element. For any such island that one can imagine, it is always possible to imagine an island that is a bit greater.

The mathematical formulation in the corollary also raises questions. One such question involves the composition of the set E of things that exist in reality: Does E include, for example, mathematical objects like numbers, sets, and theorems? One model for the ontological status of mathematical objects, proposed in chapter 3 of [3], says that mathematical objects do indeed exist as thoughts in the mind of God. In this model, mathematical objects are part of creation, and their existence is continually sustained by God because God continually thinks them.

The existence of mathematical objects suggests a second question on the difficulty of drawing comparisons between pairs of elements of the set T. How might we consider one mathematical object to be "better" than another, for example? In response, I will point out that there is no requirement that the ordering $\succ$ be able to compare every pair of elements of T. (We saw this in the example of the coordinate-wise ordering on $\mathbb{R} \times \mathbb{R}$.) The Theorem only requires that each element of T can be compared with the maximum element M, and that each element of $T \setminus E$ can be compared with some element of E.

A third question involves the plausibility of assumption 2 in the context of the corollary. This assumption has an intuitive appeal. Having a real friend, for example, should be better than having an imaginary one. For a person stranded in the desert, a real oasis is better than a mirage. A correct proof of a mathematical theorem is certainly better than an argument containing a flaw. But does adding "existence" to the description of something always improve that thing? Assumption 2 has been a controversial part of the ontological argument.

This note merely scratches the surface of a fascinating topic. For further discussion of the ontological

[1]Analogous statements made about Jesus in the New Testament (Rev 22:13; Phil 2:9-11) are thus affirmations of the deity of Jesus; see for example [1].

argument, see for example the fourth chapter of [2], the third chapter of [5], and [4]. Anselm went on to argue that not only does God exist, but God must in fact exist necessarily—that is, God exists in every conceivable world. Modern discussions of this form of Anselm's argument often use *modal logics* that distinguish between possible and necessary existence.

I formulated the theorem in this note as a means of better understanding Anselm's original argument. This theorem also might an interesting addition to a mathematics class that discusses relations and their properties.

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References

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